Number Sense and Number Nonsense

Understanding the Challenges of Learning Math

by

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and

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Sara Shunkwiler, M.Ed., taught middle school at Marburn Academy, a private school in Columbus, Ohio, for bright children who learn differently. She is currently teaching Pre-Algebra and Algebra in a public school setting and has worked with many students who have varying degrees of math difficulty. Prior to entering teaching, she was an engineer, a career she chose when she placed third in a schoolwide algebra contest and discovered she was "good at math." Her team won the state competition, and the three women on that team went on to graduate at the top of their engineering classes at The Ohio State University. Ms. Shunkwiler also earned a master of science degree in ceramic engineering from the University of Illinois at Urbana-Champaign. For 12 years, she worked as a product development and test engineer with General Motors and received three United States patents during that time. She left engineering to share her love of mathematics and science with students in the pivotal middle school years, and she earned a master of education degree in middle childhood mathematics and science education from The Ohio State University. She and her husband, also a ceramic engineer, live in Frederick, Maryland, with their two teenage sons.

Preface

In the early fall a few years ago, a college senior named Abby showed up in tears to my psychology practice for a diagnostic evaluation. She was a hard-working honor student and respected peer tutor in English, but there was a good chance that she would not graduate. Why? To receive a degree from her college, she was required to pass one class of precalculus level mathematics.

Abby had struggled with math throughout her elementary years and, even though she did well in her other subjects, was barely able to earn enough math credits to graduate from high school. An exam during her college orientation had placed her into a noncredit remedial math course. After four failed attempts to pass that class, the dean referred her for an evaluation. She was frantic by the time she arrived at my office.

What was I to make of Abby? Her predicament raised many questions. Was her math difficulty a symptom of a disorder, as suggested by official psychiatric guidelines? If so, what evidence was I looking for? As a psychologist, I had seen many students who did poorly in math—in fact, many students seeking evaluations, for whatever reason, complain of struggling with math. That these students had trouble with math was indisputable; my job was to figure out *why*—without knowing the source of the problem, there would be no way to fix it. Psychiatric guidelines suggest that various cognitive impairments can be involved. Indeed, each student, including Abby, demonstrated a unique set of cognitive strengths and weaknesses. There was no pattern in their cognitive test results that might explain what they all had in common, which was math failure. Nor was there any clear way to understand how their individual cognitive weaknesses contributed to this shared outcome.

I also wondered whether Abby's plight was qualitatively different from other students' math challenges. Perhaps it was just the extreme end of a continuum that embraced any number of humanities majors who were never referred for evaluation simply because they were not required to take math. Many people, like Abby, do not like numbers and, when possible, avoid balancing their checkbooks, calculating discounts, and doubling recipes. And what about younger students who struggle with basic math until they are allowed to drop it, or who still need a calculator to reckon 4×6, even after years of flashcards? How prevalent are such difficulties? Some evidence suggests math troubles are pervasive: A recent Google search of the phrase *I suck at math* turned up 53,000 hits, mostly message-board math queries. How could I help Abby and others like her make sense of their math difficulties, and what advice could I give to their teachers and professors?

And so my quest began. One math curriculum expert told me that there was little research on math learning disabilities, and in a strict sense she was right. A few large-scale studies have shown, for example, that some students with math difficulties also have a reading disorder, but that many do not, and the few existing neuropsychological studies have been inconclusive; that was about all the available information. Moreover, one prominent researcher told me that science was not close to producing assessment or educational guidelines. Indeed, in that regard, research on math impairments is at least a decade behind that on dyslexia.

It quickly became clear that to understand why some people fail in math, I first had to understand what enables most people to succeed. That is, what cognitive skills are necessary for learning basic mathematics? On this topic, as it turns out, there is abundant research. Indeed, reviewing it has been, to borrow a phrase from Geekspeak, a bit like drinking from a fire hose. Scientists in animal and infant behavior, cognitive psychology and development, education, linguistics, genetics, neuropsychology, and most recently, neuroscience have all made preliminary though significant contributions to the field. Unfortunately, the research has largely remained buried in the journals of the individual academic disciplines; consequently, these findings have been inaccessible to educators, psychologists, and even to researchers across fields. Those results that have seen daylight have emerged in books for the general reader who is curious about the mind and mathematics, in texts restricted to arithmetic development in very young children, and more recently, in a few scholarly essay collections. For this reason, I decided to write a book that would integrate, for the first time, this vast body of work for practitioners who have a stake in its contributions.

How one learns math, however, depends on what happens in the classroom, as well as on what happens in the mind. Solving that piece of the puzzle would require the perspective and insights of a teacher both well versed in math and mathematical pedagogy and experienced in teaching students with learning differences. Hence, I recruited Sara Shunkwiler to join the project. A middle school math teacher with a special interest in children with learning disabilities, she knew first-hand the frustration of teaching students who, despite their hard work and her own dedication, simply could not master even the most basic material. She readily accepted my invitation; in particular, she was eager to make the psychological insights found in the research accessible to the classroom teacher and to address several broad, pertinent educational issues. To this end, she wrote the book's final chapter and additionally provided invaluable advice on the other ten.

Together, then, we aim to fill the literature gap with a book that asks the following questions: What cognitive skills are necessary for doing mathematics and how do those skills develop? Why do some people have trouble with math and how can that difficulty be evaluated? What do the scientific findings to date imply for education? Where should research go from here? We intend to answer these questions based on the latest scientific data, which in most areas are still quite preliminary.

Here are our arguments. We find that mathematics draws on three basic modes of thought and extraordinarily complex brain circuits, comprising a wide variety of related perceptual, cognitive, executive, and reasoning skills that under ideal conditions work together seamlessly. Because these systems depend on each other for full and efficient functioning, an impairment or perturbation in any part of the network can interfere significantly with the ability to learn and do mathematics. Math disability, therefore, is not unitary. Rather, it appears to result from any number of cognitive glitches, impairments, and asymmetries, often exacerbated by emotional and cultural issues or by instruction poorly matched to the student's way of thinking. For this reason, any evaluation of a student for unexpected and debilitating difficulty in learning math must be comprehensive. Math disability also exists on a continuum with ability such that the severe difficulties of some individuals are not qualitatively different from the occasional troubles experienced by the rest of us. Thus, the scientific findings pertain not just to students with severe and pervasive math disabilities but to anyone who struggles with or feels uneasy around numbers.

In addition, we argue that mathematics is difficult because the relevant cerebral networks, with one partial exception, are not inherently specific to number. Mathematics must be learned, and teachers will be most effective when they understand their students' unique intellectual strengths and weaknesses. A number of pedagogical tools and techniques already on the market and in the classroom appear compatible with the scientific evidence, but their effectiveness—for either the general classroom or for struggling students—has only recently begun to be subjected to rigorous testing. Other simple techniques have proven surprisingly effective. We review herein the handful of recent pedagogical studies.

Our goal is to make the science of mathematical cognition accessible to a variety of professionals, including both practitioners and academics. For practitioners—teachers, psychologists, and other professionals in daily contact with confused and discouraged students—we have synthesized the scientific findings in plain English, using the technical terms of the academic disciplines when necessary, but supplying definitions as well as examples and illustrations for additional clarification. By including illustrative student cases, we hope to bring the research findings to life and to convey the toll that our lack of knowledge takes on frustrated students and, by extension, on the lives of adults and on the economy and productivity of the nation.

For academics—those doing the research—we provide a wide-ranging bibliography, our gift to graduate students everywhere. Our review focuses chiefly on controlled studies, citing other work only when its insights seem especially pertinent. We alert all readers at the outset, however, that many of the studies are small and narrowly focused; most need to be replicated. Although many are excellent, some are methodologically flawed and thus should be interpreted cautiously. This book was never intended to be the last word on the subject—indeed, it is one of the very first words—and was written partly to provide researchers with a starting point for their own future work.

In attempting to write a book that would be accessible to a diverse readership, we elected to locate the scholarly apparatus in chapter endnotes. The advantage, as we see it, is to make the text more readable by removing disruptive in-text citations. Moreover, doing so left us free to cite more liberally, a benefit to academics wishing to mine the sources.

As recently as 2006, American 15-year-olds scored below average in mathematics among the world's most prosperous countries.[1] Thus this book should interest not only teachers, psychologists, other practitioners, aspiring professionals, and researchers, but also public policy officials and those whose curiosity is sparked by the national debate over how to improve American mathematics education.

Nancy Krasa

Note

[1]Organisation for Economic Co-Operation and Development, 2007, Table 6.2c.

Section I

Thinking Spatially

When most people think of mathematics, they think about arithmetic facts, equations, proofs, and the like. This math requires an ordered list of counting words and matching written numerals. It also requires other language that tells us what to do with the numbers, such as *multiply* or *take the square root,* or that describes how shapes relate to each other, such as *parallel* or *congruent.* Furthermore, it includes a long list of rules that explain how they all work together. This is the math we study in school.

But what if mathematical language and symbols did not exist? Could people think about quantity at all? Section I examines the intuitive side of mathematics and how that intuition influences people's grasp of the concepts that the conventional symbols represent. The old philosophers appreciated that mathematical intuition is intimately tied to one's sense of space and time; by envisioning objects and events in those dimensions, one comes to understand the structure of our universe, the stuff of mathematics.[1] These mental pictures help people not only to understand formal mathematics, but also to gain new mathematical insights. As Albert Einstein famously remarked,

> Words and language, whether written or spoken, do not seem to play any part in my thought processes. The psychological entities that serve as building blocks for my thought are certain signs or images, more or less clear, that I can reproduce and recombine at will.[2]

Chapter 2 explores how spatial insight affects one's understanding of number and quantitative problem solving. Chapter 3 shows how it informs geometry, map and model reading, and proportional and mechanical reasoning—significant branches and applications of mathematics. In the course of that discussion, we also examine how failing to form spatial images can compromise math achievement for some students.

Notes

[1]See Boroditsky, 2000.
[2]As cited in Dehaene, 1997, p. 151.

Chapter 2

Number Sense

Science has demonstrated that humans are not alone in the ability to quantify. Animals, which do not have "words and language, whether written or spoken," use numbers every day for survival.[1] Across the phylogenetic spectrum from insects to chimpanzees, survival of both the individual and the species depends on quantitative skills to communicate, forage, evaluate threat, track offspring, optimize breeding, and conserve energy. Animals' remarkable quantitative abilities include a sense of *how many* (e.g., eggs in the nest) and *how much* (e.g., distance from predators); some animals have even been trained to determine *which one* in a series (e.g., the third tunnel in a rat's maze). Animals make these judgments based on information obtained through all of their senses.[2]

Even though they cannot yet talk, human infants also have a rudimentary sense of quantity. For example, infants can distinguish two cookies from three and know that adding to, or taking away from, a small number of toys brings predictable results. In studying infants' mathematical skills, scholars debate two key issues. The first is whether these very young children can clearly distinguish between *how many* and *how much*. For example, in judging whether two cookies are the same as three cookies, they may be basing their decision on the amount of "cookie stuff" (i.e., the surface area or volume of the cookies) or on the cookie numerosity (i.e., the countable number of cookies). (A note about terminology: In this book, we use various terms for number. *Numerosity* refers to the amount of discrete things or events in a collection and is a property of the collection itself. *Numeral* is the written symbol. We use *number* generically, in the singular or plural, to denote discrete quantity regardless of format.)

Unfortunately, infants are too young to tell us what they are thinking. So far, research has been inconclusive and the debate remains lively. For some scientists, the resolution of this first issue hinges on the second key question: whether human quantitative abilities are directly inherited from animals through evolution or rather represent a specifically human adaptation of more basic shared perceptual skills. Scholars continue to debate this thorny question as well.[3]

The Mental Number Line

Researchers do agree, however, that both animals and young humans have a rudimentary sense of quantity. These primitive quantitative notions have two striking qualities: They are relative and approximate. Without being able to count, both animals and human infants judge quantities in relation to other quantities. For example, two cookies take on quantitative meaning only in relation to one cookie (i.e., more) or three cookies (i.e., less). Many animals live or die based on their skill at determining *more*—pertaining not just to continuous amounts such as water, distance, or time, but also to discrete amounts of food bits, predators, and eggs. Indeed, many creatures can weigh one factor against another in a truly remarkable cost-benefit analysis.

Young children first demonstrate a rudimentary, implicit sense of *more* around the end of their first year or early in their second year, when they pick three cookies over two. During their third year, when they can purposefully manipulate objects and understand instructions (but before they can count reliably), children make explicit ordinal judgments (e.g., picking the "winner," or larger, of two small collections of boxes). By age 5 years, children can compare numerosities from memory, suggesting they have a mental representation or image of them.[4]

Because humans' early idea of numerosity is relative and therefore ordered, scholars posit that the concept of numerosity is fundamentally spatial—a mental number line on which values are envisioned from small to large, much as a ruler shows distance. Once that mental landscape is established, one can determine relative value by comparing locations on the imaginary line. In this manner, one can mentally record and remember that 2 items are a bit fewer than 3 items and that 6 items are a lot more, regardless of the items. Distance along the line becomes a mental analog for abstract number, much as an analog clock depicts time.

In one respect, however, this early, precounting mental number line does not resemble a ruler; this difference has to do with the way people think about quantity when they do not count. For example, without counting, one cannot determine exactly how many birds are in Figure 2.1; at best one can match up the flocks, bird for bird, to see whether any birds are left over—a time-consuming and impractical process. Thus the second key feature of primitive quantification is that it is approximate.

If the uncounted comparisons cannot be precise, how accurate are they likely to be? To answer that question, we return to Figure 2.1: Which decision was easier, A versus B or B versus C? Most people will find the first comparison to be easier. Approximate comparisons are governed by a psychophysical principle called Weber's law. According to Weber's law, accuracy depends not only on the size of the values, but also on their difference: The closer two values are to each other, the harder it is to tell them apart. In other words, the ability to approximately distinguish two values from each other depends upon their ratio: the more similar the values (i.e., the closer their ratio is to 1), the more difficult the distinction. Flocks A and B have 3 and 6 birds, respectively; thus their ratio is 1:2. Flock C has 7 birds, making the ratio of flock B to flock C equal to 6:7—a value much closer to 1. Therefore, distinguish-

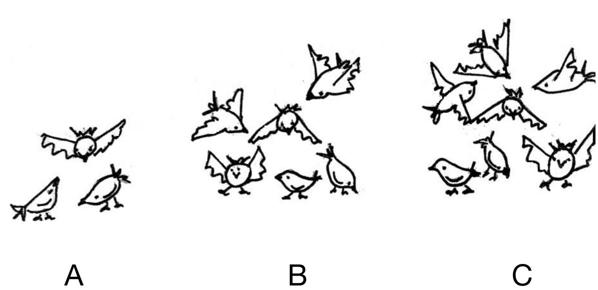


Figure 2.1 Without counting, can you tell whether Flock A is the same size as Flock B? What About Flock B and Flock C?

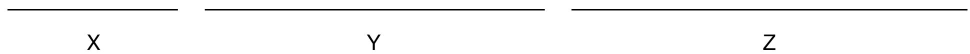


Figure 2.2. Without measuring, can you tell which line is longer, X or Y? What about Y versus Z?

ing 6 birds from 7 birds is more difficult than distinguishing 3 birds from 6 birds, when counting is not an option. The larger the collections, the greater the disparity between them must be for someone to detect a difference without counting. If a 6-bird flock lost 3 members, one would notice the difference; if a 100-bird flock lost 3 members, it would be undetectable. (By contrast, errors in exact counting arise when one loses track; the higher the numerosity, the greater the chance of making an error. Thus, exact counting error is simply and directly related to numerosity, not to a ratio of numerosities.)[5]

Weber's law applies to any sequential ordering along a discrete or continuous dimension, such as the alphabet or linear distance. For example, without counting, consider which letter is closer to the letter *O* in the alphabet: *L* or *B*? What about *M* versus *R*? Regarding Figure 2.2, which line is longer? Is it X or Y? How about Y versus Z? Weber's law affects how people compare all kinds of ordered things: the months of the year, weight, color, musical pitch, and numerosity.[6]

As children amass, sort, and distribute collections of things during the toddler, preschool, and kindergarten years, they realize that the sizes of their collections vary and develop a subjective impression of ordered numerosity. Unlike the usual, evenly spaced (i.e., linear) number line that looks like a ruler, this earlier mental image reflects children's greater familiarity with the small values that they use frequently, know well, and can envision sharply and distinctly. By contrast, seldom-used larger quantities are much murkier to young children, as it is harder to tell the difference between two large sets of items if one cannot count them. Larger and less familiar quantities seem less distinct and are therefore more difficult to compare. Preschoolers, for example, can distinguish four items from two but not from six items. A set much larger than about five just seems like "a lot" to a preschooler; finer distinctions are not yet possible. Thus, the early subjective mental number line looks peculiar, with low values spread out at one end and the larger ones bunched up indistinguishably at the other end. Mathematically speaking, these values are arranged more or less logarithmically rather than linearly, as shown in Figure 2.3.[7]

Saul Steinberg's classic "View of the World from 9th Avenue" imaginatively illustrates this youthful mental number line. It depicts a myopic New Yorker's perspective looking west: The artist renders the familiar neighborhood along New York City's Ninth and Tenth Avenues in great detail, much the way people clearly

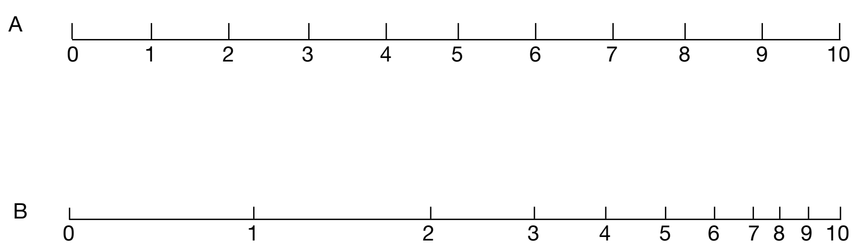


Figure 2.3. Number lines. A) Linear. B) Logarithmic.

"see" the small, frequently used numbers. Meanwhile, New Jersey, the rest of the continent, the Pacific Ocean, and Asia occupy the leftover space. The view does not offer a sharp geographic distinction between other cities, although it does vaguely represent the general terrain. Similarly, large numbers are out there somewhere in a child's mind, but their exact locations are unclear.

"View of the World from 9th Avenue" © The Saul Steinberg Foundation/ Artists Rights Society (ARS), New York; reprinted by permission.

When exact enumeration is not an option, the only possible view of quantity is approximate and relative. Cultures lacking a counting system can describe and remember collections of objects only approximately, with a Weber-like error pattern mirroring that of very young children.[8] But what happens to these capabilities when people learn to count? Counting—a uniquely human faculty tied to language—allows one to enumerate and remember precisely and absolutely, and opens the door to exact mathematics. For example, by counting, one can describe and remember the exact number of birds in each flock of Figure 2.1; one can also say with certainty which flock is biggest. In some ways, then, there are two entirely different ways to understand quantity. The relationship between these two quantitative systems—the relative and approximate versus the absolute and exact—is one of the most perplexing issues for cognitive scientists.

Evidence of an embryonic mental number line is seen early in children learning how to count. Even 2½-year-olds, when shown a row of three or four objects, will add to or take away from it at one end rather than in the middle, suggesting they think of amounts as a progression.[9] Once children have mastered the basic counting principles, around age 5 or 6 years, they can compare small, familiar quantities (e.g., 2 versus 5), estimate small sums, and enter the number sequence without counting up from 1.[10] This earliest view of numbers is still highly subjective and zero-centric, however. When young children are asked to place a number on a physical number line with an end point that baffles them, they often simply count up from zero, sometimes making hatch marks as they go—just as some New Yorkers may regard the whole world from the viewpoint of Ninth Avenue. Children comprehend small, recognizable numbers, but any forays into the unfamiliar territory of higher values generally produce the errors predicted by Weber's law.

As children gain experience with larger numbers and the counting principles, their mental number line begins to look more linear and they start to manipulate numbers more accurately. In a series of studies, researchers gave pencil-and-paper number lines with only a 0 on one end and 100 on the other end to groups of kindergarten, first-grade, and second-grade students. The children were asked to mark where they thought certain numbers belonged, plotting each number on its own separate number line. The response pattern was logarithmic for the youngest children but became increasingly linear, and hence more accurate, for the older ones.[11] Second-, fourth-, and sixth-grade students produced even more dramatic results with a 1–1,000 number line. Most second-grade students and about half of the fourth-grade students produced a logarithmic response pattern, whereas the older children's estimates were robustly linear, as illustrated in Figure 2.4.[12] The investigators then wondered if the number line range might have influenced the children's accuracy. So they asked second-grade students to mark where certain numbers should go on 0–100 number lines. As predicted, their responses were roughly accurate. The children were then asked to locate those same numbers (i.e., all less than 100) on 0–1,000 number lines. In this larger, less familiar numerical neighborhood,

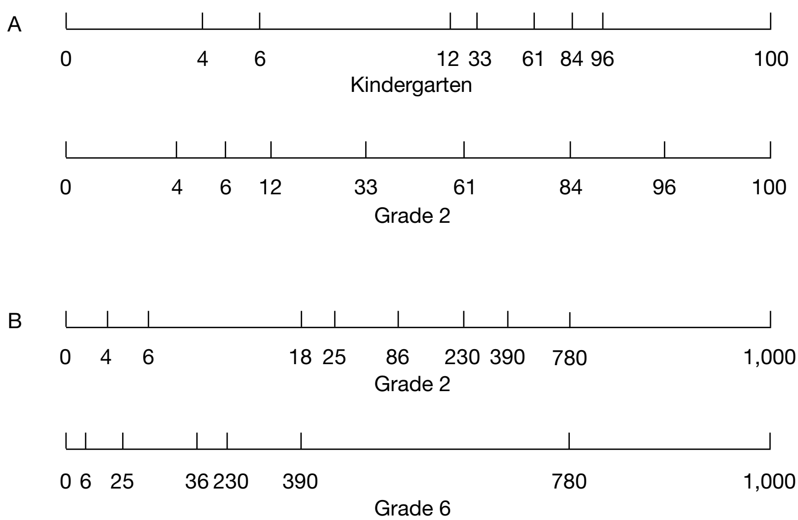


Figure 2.4. Estimated number placements. A) On a 0–100 number line, the results from kindergarten versus second-grade students. B) On a 0–1,000 number line, the results from second- versus sixth-grade students. (Sources: Siegler & Booth, 2004; Siegler & Opfer, 2003.)

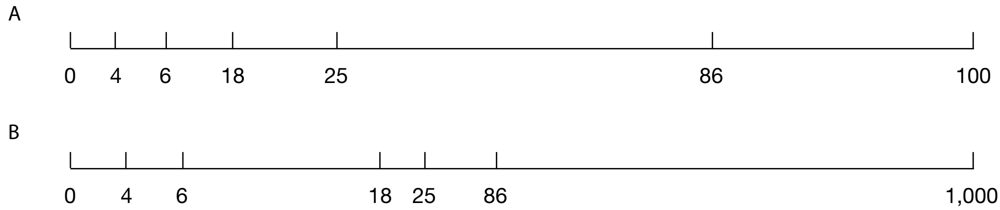


Figure 2.5. Estimated number placements of second-grade students on A) a 0–100 number line and B) a 0–1,000 number line. (*Source:* Siegler & Opfer, 2003.)

the children lost their bearings and produced estimates that assigned outsized proportions to a few relatively small numbers, as shown in Figure 2.5.[13]

In third grade, children begin to think proportionally about the number line.[14] As children learn to parse the numerical landscape and reason proportionally, they can distribute numbers more accurately between the end points of bounded number lines. They also learn to use round numbers to moor their estimations. Going back to the worldview of the myopic New Yorker, just as a few major cities and landmarks protrude from this map, the familiar round numbers (5, 10, 20, 50, 100, 1,000) stand out on the mental number line. They come into focus more clearly than the surrounding countryside and thus can be used to locate other numbers. For example, many people learn history by using key dates (1066, 1492, 1776) to orient themselves on the time line.[15] In the same way, older children and adults often use such landmarks as 250, 500, and 750 to anchor estimates on a 0–1,000 number line.[16] This manner of thinking is crucial to learning fractions, which depends on the idea that the number line can be apportioned. With no natural counting sequence, fractions derive their order and relative values from their places between 0 and 1 on the number line. By fifth or sixth grade, children can begin to track both apportionment and numerical comparison to locate fractions on a number line.[17]

Just as travel can broaden one's perspective, so children's intellectual excursion into the world of numbers can hone their quantitative sensibility, particularly through their exposure to activities related to the number line. Children's measurement estimates (e.g., "If this line is 1 inch long, draw one that is about 5 inches long"), numerosity estimates (e.g., "Guess how many candies are in the jar"), and number categorizations (e.g., "Is this number big or small?"), like their number-line placements, all start out logarithmically distorted and become more linear with age and experience. In studies, children who were good at one kind of estimation task tended to be good at the others, suggesting that estimation skills all rely on a single mental representation of quantity.[18] (Interestingly, preliminary findings suggest that girls may lag somewhat behind boys in developing number-line skills. The reason for this is not known and the results have not yet been replicated.[19])

Although familiarity with the numerical neighborhood is key to understanding mathematics, people typically develop the skill only so far. Unless one is an astronomer or works at the U.S. Office of Management and Budget, even most adults do not truly understand that 1 billion is only 1/1,000 of 1 trillion (see Figure 2.6). *A billion* and *a trillion* are simply synonymous with *a whole lot* to many people.

Figure 2.6. To many people's surprise, 1 billion is only 1/1,000 of 1 trillion.

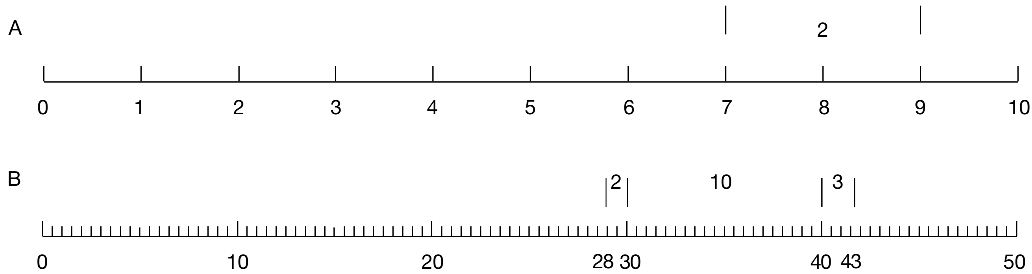


Figure 2.7. Arithmetic on the number line. A) 9 − 7 = 2. B) Decompose 15 to visualize 28 + 15 = 28 + 2 + 10 + 3 = 43, or 43 − 15 = 43 − 3 − 10 − 2 = 28.

Just as people can visualize numerical relations, so can they conjure arithmetic operations visually on a line. One can imagine arithmetic as distances between two points (subtraction) and increments along the line (addition), as illustrated in Figure 2.7, and as repeated increments (multiplication) and equal apportionment (division). Of these, subtraction translates most easily to a spatial analog because it simply involves comparing two points. Although demonstrating multiplication on a physical number line as repeated addition is a valuable teaching tool, the values quickly become too large and complex for the number line to be useful as a mental template on which to routinely conduct the operation. For this reason, children usually learn the multiplication table verbally.

Preliminary efforts to develop kindergarten math screening tests have consistently found that performance on questions of numerical comparison, on and off the number line, was one of the strongest predictors of math achievement in the first few grades.[20] Throughout the elementary years, children's math achievement has been linked to number-line skills and estimation ability in all its applications.[21] It is clear that math ability depends on one's grasp of the most fundamental concepts: numerical values and their relationships.

For most people, ideas about number mature; they learn to estimate reasonably and calculate precisely. However, that does not mean that they never again think of number subjectively. In fact, Weber's law is always lurking; people are most likely to succumb to it when they are in a hurry or cannot count or calculate. Often in daily adult life, people need to solve a problem quickly or estimate the size of a crowd; on those occasions, they tend to draw on their subjective impressions. For example, answer this quickly without calculating: Is it true that $4 + 13 \neq 60$? What about $4 + 13 \neq 19$? Most people find the first question easier to answer than the second because the target number is farther from the true sum—an example of Weber's law at work.[22]

Number and the Brain

How does the brain actually code quantitative information? Advances in imaging techniques that can map brain activation during mental activity provide insight into how the brain "knows" number. Studies of individuals engaged in mathematical thought show that the brain's surface (cortex) becomes active in part of a channel on each side (hemisphere), known as the intraparietal sulcus (IPS; Figure 2.8). The IPS (plural: intraparietal sulci) are highly sensitive to number, regardless of whether it is presented in spoken ("six") or written (*six*) word format or as a numeral (*6*). Moreover, most scholars also agree that the IPS activate in response to concrete nu-

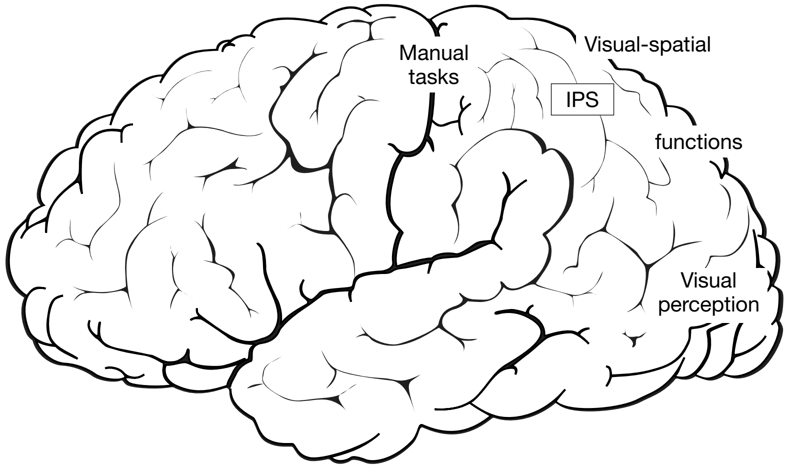


Figure 2.8. View of the brain's left hemisphere, showing the quantity-sensitive intraparietal sulcus (IPS) and surrounding visual-spatial region, as well as the areas activated by incoming visual perceptions and by manual tasks. These functions also reside in the right hemisphere. (*Sources:* Ansari, 2008; Jordan, Wüstenberg, Heinze, Peters, & Jancke, 2002; Simon, Mangin, Cohen, LeBihan, & Dehaene, 2002.)

merosity (******) (e.g., when people compare arrays of dots without counting). As discussed in Chapter 6, the IPS become more sensitive to number as children develop. The IPS encode number only approximately, however, and have difficulty distinguishing two values with a ratio close to 1.[23] The IPS are most active when people compare or estimate quantities, subtract one quantity from another, find the midpoint between two numbers, judge the relative proximity of two values to a third value, and manipulate quantities—that is, when people engage in tasks that can be easily envisioned on a number line.[24]

Significantly, the IPS are embedded in brain regions dedicated to a wide variety of visual-spatial functions. These regions are part of the spatial (i.e., "where") visual pathway connecting visual perception to muscle activities, particularly those involving the hands, and processing information about objects' location.[25] (The "what" visual pathway, also variously referred to as the iconic or object pathway, by contrast, specializes in identifying objects, as well as distinguishing color and shape, and is routed elsewhere.) Difficulty with quantitative manipulations is linked to various medical conditions, such as Gerstmann and Turner syndromes, characterized by damage in this region.[26] Preliminary behavioral data from research in progress suggests a likely association between spatial skill and number-line placement accuracy among typically developing primary-grade students.[27] Moreover, as discussed in Chapter 6, IPS abnormalities have been linked to math learning disabilities in children.[28]

Indeed, several behavioral studies of individuals with brain injuries strongly suggest that people think of numbers and their relationships in visual-spatial terms, as if they could actually see them lined up from small to large along a real line in their minds. For example, one study examined individuals with right-hemisphere brain injuries resulting in hemianopia. In this condition, people lose awareness of their left visual field; when asked to draw figures, they tend to minimize or omit the parts that would be seen to the left of center. The individuals in this study revealed a marked inability to name the midpoint between two numbers (e.g., naming the midpoint between 11 and 19 as 17), even when they did the problem without paper or pencil and responded orally. That is, they committed mental errors similar to the

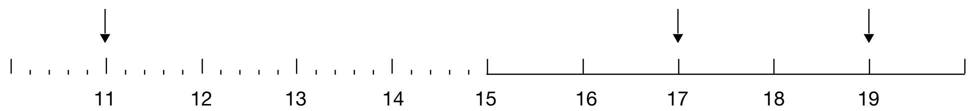


Figure 2.9. A patient with hemianopia responded, "Seventeen," when asked, "What is the midpoint between eleven and nineteen?" (*Source:* Zorzi, Priftis, & Umiltà, 2002.)

more concrete ones such individuals typically make when asked to mark the midpoint of a horizontal line on paper.[29] Figure 2.9 illustrates what the verbal error of an individual with hemianopia might look like on paper and, presumably, in the mind.

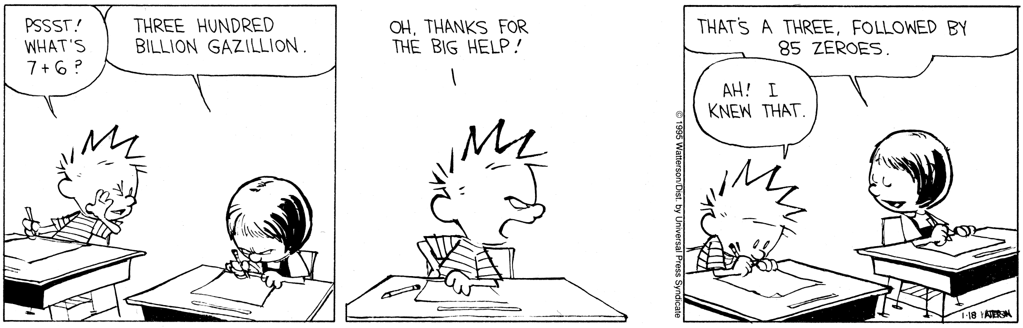
It may seem that all spatial sense is necessarily tied to vision, and research often conflates the terms *spatial* and *visual-spatial.* The distinction between them with regard to number sense is not clear, however. A potentially fruitful line of research on the role of purely spatial ability would be to investigate the mental representation of number in individuals who were born blind. One aspect of congenital blindness is that the knowledge of quantity is restricted to sequential auditory and tactile input with the consequent additional memory burden, or to small numerosities of handheld objects. Young blind children do not use their fingers for counting, but instead use a double-counting system when counting aloud and estimate the results of simple calculations based on a sense of "many-ness" obtained tactilely.[30] More research into vision's relevance to the development of number sense is warranted.

Number Sense and Number Nonsense

Through instruction and experience with quantities, children become familiar with numbers and develop a reliable mental picture of how they relate to each other. A firm grasp of relative quantity fosters more varied problem-solving strategies, more courage to estimate, better judgment about the reasonableness of a solution, and even easier fact retrieval.[31] This confidence with quantities and their mental manipulation can be regarded as *number sense.* As psychologist Ann Dowker remarked,

> To the person without number sense, arithmetic is a bewildering territory in which any deviation from the known path may rapidly lead to being totally lost. The person with number sense . . . has, metaphorically, an effective "cognitive map" of that same territory, which means such deviations can be tolerated, since the person can expect to be able to correct them if they cause problems and is unlikely to become lost in any serious sense.[32]

Most—but not all—children bring an intuitive quantitative sense with them on their first day of kindergarten.[33] For many of those who do start out with a rudimentary grasp of quantity, however, digits often become disembodied from their values and technical arithmetic skills become unmoored from number sense in the course of conventional mathematical education. Sometimes, students rediscover their number sense through computational simplification, like rounding, late in elementary school.[34] However, some students never regain their quantitative intuition and continue through life performing calculations mechanically, without any real idea of what the computations mean. Other students move easily between exact calculations and intuitive approximations, enabling them to tackle unconventional problems, use time-saving short cuts, and devise creative solutions. (In this regard, boys



CALVIN AND HOBBES © 1995 Watterson. Dist. by UNIVERSAL PRESS SYNDICATE. Reprinted with permission. All rights reserved.

on average again seem to have an advantage, at least in adolescence; psychologists are trying to find out why.[35])

Like hapless Calvin, some people either lack number sense or do not use it; in fact, most people seem to favor exact—but often poorly understood—numbers.[36] Businesses understand this concept (indeed, they bank on it) when they price $10 items at $9.95, knowing that many customers will be oblivious to the proximity of the latter price to the former. Weak understanding of what numbers actually mean and how they relate to each other—call it *number nonsense*—creates a disadvantage at school and in life when one must verify that calculation results are reasonable or understand a computation's implications.

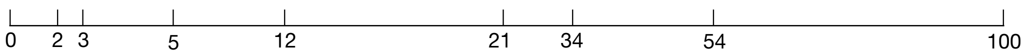
When children fail to develop number sense, it may lead to other serious math problems. Studies of primary-grade students with severe math impairment found that they were slow to make numerical comparisons (e.g., answering "Which is bigger, five or four?") or made immature number-line placements—both hallmarks of poor number sense. Their slow recital of the counting sequence also suggested a weak grasp of number order, and many had significant difficulty on all other number tasks as well.[37] Tenuous number sense may explain why some young students fail to add using the minimum addend strategy, a common early addition method whereby students determine the larger addend and count up by the smaller term (e.g., 6 + 3 = "six . . . seven, eight, nine"). These students may simply not be able to decide which number is the bigger addend or to enter the number sequence at any place other than "one." Because of these difficulties, many such students also fail to master arithmetic facts.[38]

Whereas number sense is closely related to spatial skill, it appears to be quite distinct from some other cognitive skills, such that impairments in one do not imply impairments in the other. For example, some children with receptive and expressive language impairments or dyslexia are able to compare values even if they cannot name or read them, presumably because they understand the relation of one quantity to another.[39] Thus, having a viable mental number line may be regarded as a cognitive function at least partially independent of some other learning-related skills, and compromised number sense may account for some children's math learning difficulties. In fact, severe mathematical disability with no other learning difficulties is not unusual. Most large-scale studies have identified groups of students who have math impairments but no other significant learning disabilities. Future behavioral and neurocognitive research promises to further elucidate the relation between some mathematical disability and impairments in number sense, in the neural circuits that support it, and perhaps in spatial skill.

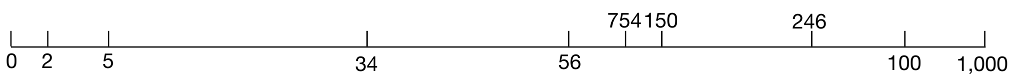
Difficulty with number lines

Emily is a lively 13-year-old with an outgoing personality. Her birth and early development were typical and milestones were on schedule. Now, in seventh grade, her math teacher reports that she seems to lack sense of what numbers mean. She calculates inefficiently and entirely by rote without understanding what she is doing or why.

When asked to place a series of numbers on individual 0–100 number lines, Emily showed the perspective on numbers typically seen in kindergartners and first graders, in which small numbers seem bigger than they are.



Emily is even less at home in the world of larger numbers, with which seventh graders should be familiar. In locating numbers between 0 and 1,000, she nearly filled the range with numbers less than 100, and she was baffled about the relative values of numbers greater than 100.



Without a reliable, realistic mental representation of ordered quantity, Emily is left with only rote procedures, making mental math especially challenging. For example, when asked (in language she could understand) to mentally subtract 3 serially, beginning at 35, she slowly responded, "Thirty-five, thirty-two, thirty, thirty-six, thirty-three, zero."

Emily received very little prior math instruction using the number line. She learned to calculate using a form of counting, and her teacher's efforts to get Emily to try another method have been unsuccessful. Studies of the effectiveness of intensive number-line instruction to establish a reliable mental number line have focused on much younger children. Testing revealed that Emily's spatial skills are adequate, so it is possible that she would benefit from such instruction.

Classroom Implications

The most difficult aspect of teaching math to young children is to keep their number sense alive and to foster a connection between it and conventional mathematics. Because number sense seems to depend on a reliable mental number line, researchers are now exploring concrete number-line activities, which represent numbers physically as they are represented mentally, as a potentially effective way to teach students about number.

Number lines are not new to the classroom. Teachers have long used number lines created from common objects, including pencil and paper, plastic strips, cardboard tubing, bead strings, linear numerical board games, rulers, calendars, and thermometers—not to mention the chalkboard number line at the front of the classroom. One popular device consists of rectangular sticks of graduated lengths—referred to here generically as *number blocks*—that can be lined up along a number line or track. More recently, computerized versions of these activities have become available.[40]

Like discrete manipulatives, such as buttons or beans, these linear analogs teach cardinality, ordinality, and equality; they can be introduced in tandem with conventional notation and algorithms. Unlike buttons and beans, however, a number line provides fixed visual images (analogs) for potentially unlimited values and is ordered and systematic; moreover, it represents number the way the brain does, linking counting to linear measurement.[41] Number lines can also illustrate the conceptual underpinnings of nearly all elementary number skills, including inequality, numerical comparisons, arithmetic transformations, fractions, decimals, measurement, and negative numbers. Buttons and beans represent quantity as we most often encounter it in life, as random collections; the number line provides a schematic image or mental template that children can rely on and abstract from.

Research with young students suggests that employing number lines in the classroom can significantly improve numerical understanding. For example, helping kindergartners sort the numerals 1–100 into equal-sized piles of *very small, small, medium, big,* and *very big* numbers effectively provided the students with a linear (and thus more accurate) sense of numbers in that range.[42] First-grade students improved their understanding of missing-addend problems using number-line based instruction incorporating practice and feedback.[43] First-grade students also improved in both number-line accuracy and addition skills using computerized number-line illustrations of addends and sums.[44] Instruction using unmarked number lines enabled third-grade students to develop both flexible strategies and procedural competence with multidigit addition and subtraction.[45] Simply correcting second-grade students' most discrepant number placement (typically 150 on the 0–1,000 number line) produced particularly dramatic recalibration.[46] (Researchers do not yet know whether this technique is as effective with children whose view of number is significantly impaired.) Teaching college students to diagram analog-friendly word problems with number lines significantly improved the students' accuracy—more than simply rewording the problems did—and even improved performance on more complex problems.[47] Unfortunately, we have not yet seen any research on the effectiveness of number line pedagogy for children in the intermediate, middle, or secondary grades; such investigations are certainly warranted.

Many preschool and kindergarten children from low-income families without access to certain types of board and other number games have difficulty answering questions such as, "Which is bigger: five or four?"—a quantitative task that poses little trouble for other children of the same age. For these children, number-line activities such as linear numerical board games, which help them link the quantities indicated by spaces counted along a line, dots on dice, numbers on spinners or cards, distance moved, and duration of play have proven particularly useful. Researchers caution, however, that children should count out their moves according to the spaces on the board (e.g., "seventeen, eighteen, nineteen . . . "), not according to the value on the dice or spinner (e.g., "one, two, three . . . "), to connect the counting sequence to the number line.[48]

In one study, four 15-minute sessions using a simple linear numerical board game brought one group of underprivileged preschoolers up to the level of their middle-class peers in terms of number-line estimation, magnitude comparison, counting, and numeral identification—gains that remained 9 weeks later.[49] Another research-based program for children in prekindergarten through second grade, focusing on integrating traditional object counting with diverse child-friendly number-line activities, met with similar success. In addition to numerical board games, these activities included using thermometers and counting off as children queued up for recess—simple daily classroom activities that teachers can implement easily.[50] Explicit number-line instruction led to significantly greater improvements than did nonnumerical board games, counting activities such as card games, prereading intervention, textbook number-line illustrations without instruction or feedback, or traditional math lessons that did not use number-line activities or stress relative magnitude. In short, simple number-line techniques have proven exceptionally fruitful for young children at risk of math difficulties. Alert teachers can find additional linear counting activities in many common settings (e.g., counting steps in flights of stairs or rungs traversed on the monkey bars; playing hopscotch).

The success of early number-line instruction suggests that interventions based on number-line thinking may also prove effective in keeping children's number sense alive while they master arithmetic algorithms. For example, when a child forgets to carry (e.g., 19 + 6 = 15), a leading conceptual question (e.g., "When you add something to 19, would you expect the answer to be bigger or smaller than 19?") may be more useful than an admonition focused on procedure (e.g., "Don't forget to carry!" or "Check your work!"). We have not seen any rigorous investigation of the effectiveness of such number-sense interventions versus those oriented to arithmetic procedures; such research would provide useful guidance for teachers.

One function for which a number-line approach has proven particularly useful is big-number subtraction, where it helps students avoid some of the pitfalls associated with using the algorithm.[51] For example, when asked to mentally subtract 3 serially from 35, a student struggling to use the algorithm might respond "thirty-five, thirty-two, thirty-nine, thirty-six, thirty-three . . . ." In contrast, the transition across 30 poses less of a stumbling block if the student can envision the subtraction on a number line.

Earlier, it was noted that some students rediscover number sense when they learn rounding rules and computational estimation late in elementary school. Computational estimation must rely heavily on number sense if it is to be useful in daily life. Many people do not estimate very well because it is a complicated task involving approximating a set of numbers, keeping track of all the estimates, doing mental arithmetic with them, and adjusting the results to compensate for the simplifications. Most students take a long time to master this mental juggling act, often well into high school.[52] Nevertheless, one of the key components of this skill—a thorough familiarity with numbers and their number-line neighborhoods—is accessible to much younger children. The typical classroom expectation is for all answers to be precise and for rounding to follow certain rules, but that often causes many cautious students to be less tolerant of approximations as they get older. Conversely, if students develop an intimate knowledge of the number line at a young age, they may be ready for the more complicated estimation tasks later. For example, students can learn to recognize "friendly neighbors"—numbers they can call on, such as 20 or 25, when unwieldy numbers such as 23 are giving them trouble. Further research on the effect of early number-line instruction on downstream computations is certainly warranted.

Although number-line instruction looks very promising, many pedagogical questions remain unanswered. For example, little is known about the relative effectiveness of manipulatives, such as cardboard tubing, versus paper-and-pencil number lines versus computer software engineered to illustrate the same lessons. Each approach has advantages: Objects feature a tactile component, software permits interactive and speeded exercises designed to foster skill fluency, and paper and pencils are cheap and accessible. The relative effectiveness of marked versus unmarked number lines is also unknown, although preliminary research suggests that unmarked number lines may be more engaging and foster development of more variable strategies, even for students with impaired math skills.[53] The effect of color (an iconic feature) should also be tested: When number blocks are color coded, as most are, are students distracted from mastering the necessary association between number and length? Which number-line direction is more effective, vertical or horizontal? Do students have trouble switching from one to the other? Does the mental number line extend to negative numbers? What is the most effective way to teach negative numbers?[54] These questions remain largely unanswered.

Because number-line activities are chiefly nonverbal, some learning disability experts tout their particular usefulness in teaching students with language and reading disabilities; however, that claim has not been scientifically verified. More research is also needed on whether and how number-line instruction might be effective with older students and students whose impairments are spatial or visual rather than linguistic. In general, further research is warranted on the effectiveness of number-line instruction at all grade and ability levels and for students with a variety of cognitive profiles.

Conclusion

Long before most children see the inside of a formal classroom, they know something about number. They know that three cookies are more than two cookies and that if someone takes one cookie away, there will be fewer. They have vague ideas about *a lot* and *a little* that become sharper as they get older. As children learn to count and gain experience with numbers, they develop a mental image of how quantities relate to each other: a mental number line on which each number has its place, like inches on a ruler. Indeed, quantities are coded in the region of the brain that specializes in spatial functions; knowledge of number is intimately tied to that spatial sense.

Many children develop an easy familiarity with quantity—*number sense*—as they gain mathematical experience. With a reliable mental number line, they have a cognitive map that keeps them oriented as they wander through the unfamiliar terrain of school mathematics. Some children, however, lose their early comfort with quantity as they struggle to master arithmetic rules and procedures; others seem never to develop this comfort with numbers at all. It is not yet known why this is so, but research suggests that spatial difficulties may contribute to some children's number problems. For young children at risk for math failure, classroom number-line activities have demonstrated improvement in number sense. Chapter 3 continues the investigation of the relationship between spatial skill and math, looking beyond number to other mathematical branches and applications: geometry, way finding, and proportional and mechanical reasoning.

Notes

[1]Albert Einstein, as cited in Dehaene, 1997, p. 151.

[2]See, e.g., Andersson, 2003; Davis & Pérusse, 1988; Devenport, Patterson, & Devenport, 2005; McComb, Parker, & Pusey, 1994.

[3]See, e.g., Mix, Huttenlocher, & Levine, 2002a; Simon, 1997; Wynn, 1998.
[4]See, e.g., Barth, LaMont, Lipton, & Spelke, 2005; Brannon & Van de Walle, 2001; Mix, Huttenlocher, & Levine, 2002b.
[5]See, e.g., Dehaene, 1997, pp. 64–88.
[6]See Barth, Kanwisher, & Spelke, 2003; Gevers, Reynvoet, & Fias, 2003; Holloway & Ansari, 2008; Moyer & Landauer, 1967.
[7]Huntley-Fenner & Cannon, 2000; Siegler & Booth, 2005.
[8]Frank, Everett, Fedorenko, & Gibson, 2008.
[9]Opfer & Thompson, 2006.
[10]See, e.g., Dowker, 1997; Resnick, 1983.
[11]Siegler & Booth, 2004.
[12]Siegler & Opfer, 2003.
[13]Siegler & Opfer, 2003. See also Ebersbach, Luwel, Frick, Onghena, & Verschaffel, 2008.
[14]Petitto, 1990; Siegler & Opfer, 2003.
[15]See Dehaene, 1997, pp. 108–117.
[16]Siegler & Opfer, 2003.
[17]Rittle-Johnson, Siegler, & Alibali, 2001.
[18]Laski & Siegler, 2007.
[19]Jordan, Kaplan, Locuniak, & Ramineni, 2007; Thompson & Opfer, 2008; J. Opfer, personal communication, February 25, 2008. See also Mills, Ablard, & Stumpf, 1993.
[20]Chard et al., 2005; Jordan et al., 2007; Mazzocco & Thompson, 2005.
[21]Booth & Siegler, 2006.
[22]Ashcraft & Fierman, 1982; Ashcraft & Stazyk, 1981; Booth & Siegler, 2006; Hobbs & Kreinovich, 2006.
[23]See, e.g., Ansari & Dhital, 2006; Cantlon, Brannon, Carter, & Pelphrey, 2006; Kadosh, Kadosh, Kaas, Henik, & Goebel, 2007; Piazza, Pinel, LeBihan, & Dehaene, 2007. But see Shuman & Kanwisher, 2004.
[24]See Dehaene, 2000.
[25]See Culham & Kanwisher, 2001; Jordan, Wüstenberg, Heinze, Peters, & Jancke, 2002; Simon, Mangin, Cohen, LeBihan, & Dehaene, 2002.
[26]See, e.g., Lemer, Dehaene, Spelke, & Cohen, 2003; Molko et al., 2003.
[27]Opfer, Young, & Krasa, 2008.
[28]Price, Holloway, Räsänen, Vesterinen, & Ansari, 2007; Rotzer et al., 2008.
[29]See, e.g., Zorzi, Priftis, & Umiltà, 2002; see also Göbel, Calabria, Farné, & Rossetti, 2006.
[30]Ahlberg & Csocsán, 1999, p. 54.
[31]Canobi, Reeve, & Pattison, 1998.
[32]Dowker, 1992, p. 52.
[33]Jordan, Huttenlocher, & Levine, 1992.
[34]See LeFevre, Greenham, & Waheed, 1993.
[35]Gallagher et al., 2000.
[36]See Hanson & Hogan, 2000; LeFevre et al., 1993.
[37]Geary, Hoard, Nugent, & Byrd-Craven, 2008; Landerl, Bevan, & Butterworth, 2004.
[38]Griffin, Case, & Siegler, 1994; Robinson, Menchetti, & Torgesen, 2002.
[39]Donlan & Gourlay, 1999; Landerl et al., 2004.
[40]See Stern & Stern, 1971; Stevens & Schwartz, 2006; Wilson, Dehaene, Pinel, Revkin, Cohen, & Cohen, 2006.
[41]Stern & Stern, 1971; Griffin, 2004.
[42]Laski & Siegler, 2007.
[43]Fueyo & Bushell, 1998.
[44]Booth & Siegler, 2008.
[45]Klein, Beishuizen, & Treffers, 1998.
[46]Opfer & Siegler, 2007.
[47]Lewis, 1989.
[48]Siegler & Ramani, 2007.

[49]Ramani & Siegler, 2008.
[50]Griffin, 2004.
[51]Blöte, Van der Burg, & Klein, 2001.
[52]Sowder, 1992.
[53]Gravemeijer, 1994.
[54]Fischer, 2003.