TEACHING MATHEMATICS MEANINGFULLY

Solutions for Reaching Struggling Learners

SECOND EDITION

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Teaching Mathematics Meaningfully

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Second Edition

by

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Dr. Lovin began her career teaching mathematics to middle and high school students before making the transition to Pre-K through Grade 8. For over 20 years, she has worked in elementary and middle school classrooms. Then and now, Dr. Lovin engages with teachers in professional development as they implement a student-centered approach to teaching mathematics. At the time of this publication, she focused her research concerning teachers' mathematical knowledge for teaching on the developmental nature of prospective teachers' fraction knowledge. She has published articles in *Teaching Children Mathematics, Mathematics Teaching in the Middle School, Teaching Exceptional Children*, and the *Journal of Mathematics Teacher Education*. She coauthored the *Teaching Student-Centered Mathematics Professional Development Series* with John A. van de Walle, Karen Karp, and Jenny

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Dr. van Ingen codirects the innovative and nationally recognized Urban Teacher Residency Partnership Program. In this role, she partners with Hillsborough County Public Schools' teachers and administrators to improve the learning of both elementary students and prospective elementary teachers. She also teaches courses in mathematics education and teacher preparation at the undergraduate, masters, and doctoral levels. Dr. van Ingen holds a bachelor's degree from St. Olaf College, a master of arts in teaching from the University of Tampa, and a doctoral degree from the University of South Florida. She was elected into membership in Phi Beta Kappa and was the recipient of the prestigious STaR fellowship in mathematics education. She taught mathematics for many years in urban, inclusive middle school classrooms before her work at the university level.

Dr. van Ingen's research agenda lies at the intersection of equitable mathematics education and clinically rich teacher preparation. Her research interests include teachers' use of research to inform practice, the use of mathematics consultations to meet the mathematics learning needs of students with exceptionalities, and the implementation of integrated STEM lessons in K–5 classrooms. She regularly publishes and presents her research to audiences who work in mathematics education, special education, and teacher preparation. She is the principal investigator and coprincipal investigator for federally funded research and is active in leadership in her professional organizations.

E

Case Study

This appendix provides a case study in which two teachers, Ms. Thompson and Mr. Hart, apply the Teaching Mathematics Meaningfully Process with their struggling learners. The teachers implement each step of the process. The contents are organized as follows:

- Purpose
- Meet Ms. Thompson, Mr. Hart, and Their Students
- Identify and Understand the Mathematics
- Continuously Assess Students
- Determine Students' Math-Specific Learning Needs
- Determine Struggling Learners' Specific Learning Needs
- Plan and Implement Responsive Instruction
- Take Action

PURPOSE

The purpose of this case study is to provide you with a way to visualize how two teachers, an elementary general education math teacher and a special education teacher, might work collaboratively to utilize the Teaching Mathematics Meaningfully Process. We first introduce you to the teachers and their students. Then, we describe how the two teachers implement each of the five components of the process. Our goal is to provide you with an applied context for making sense of this process and to illustrate the types of decision making that will help you design instruction and interventions that are responsive to your students' needs.

Throughout the case study, marginal icons are included to indicate activities and decisions that illustrate specific Essential Instructional Approaches (EIAs), National Council of Teachers of Mathematics (NCTM) Effective Mathematics Teaching Practices (MTPs), and anchors for intensifying instruction within multi-tiered systems of supports/response to intervention (MTSS/RTI). Each icon includes a number indicating which EIA, MTP, or anchor for intensifying instruction that it denotes. A key to these icons is provided next. Use the icons as you read to see how various elements of instruction discussed throughout the book are applied and integrated within the teachers' classroom planning and instruction.

Key	
Essential Instructional Approaches (EIAs) EIA	1. Teach systematically. 2. Develop and explicitly share learning intentions. 3. Make instructional decisions that are student-centered and based on meaningful data. 4. Teach mathematical fluency. 5. Teach the language of mathematics through vocabulary development and discourse. 6. Provide many response opportunities with feedback. 7. Emphasize use of mathematical practices. 8. Utilize visuals. 9. Use different appropriate grouping structures. 10. Teach students to be strategic in their approach to mathematics. 11. Situate mathematics within meaningful contexts that help students to develop abstract reasoning.
NCTM (2014b) Effective Mathematics Teaching Practices (MTPs) MTP	1. Establish mathematics goals to focus learning 2. Implement tasks that promote reasoning and problem solving 3. Use and connect mathematical representations 4. Facilitate meaningful mathematical discourse 5. Pose purposeful questions 6. Build procedural fluency from conceptual understanding 7. Support productive struggle in learning mathematics 8. Elicit and use evidence of student thinking
Anchors for Intensifying Instruction within MTSS/RTI (IAs) IA	1. Purposeful Content Focus 2. Formative Assessment—Identifying What Students Know, Don't Know, and Why 3. Explicitness and Teacher Direction 4. Teach Math Metacognition 5. Opportunities to Respond 6. Amount of Time 7. Teacher–Student Ratio

MEET MS. THOMPSON, MR. HART, AND THEIR STUDENTS

Ms. Thompson is an elementary general education teacher who teaches fifth grade. She has been teaching elementary school students for 8 years; she spent 5 of those years teaching fourth and fifth grade. She has 22 students (10 boys and 12 girls) in her class. Fifteen are white/Caucasian, five are African American, one is Mexican American, and one is Korean American. Ms. Thompson's class includes five students identified as having disabilities. Four are identified as having learning disabilities and receive special education services through the Individuals with Disabilities Education Improvement Act (IDEA) of 2004 (PL 108-446). One is identified as having attention-deficit/hyperactivity disorder (ADHD) and is supported through a Section 504 accommodation plan. In general, the students not identified with disabilities perform at grade level or above. Three additional students, who are not identified as having disabilities, sometimes struggle with math and reading. Two of these students are English language learners.

Mr. Hart is a special education teacher who works with Ms. Thompson in a consultation and facilitation role, helping her support the needs of her students with disabilities, particularly in reading and mathematics. Mr. Hart co-teaches with Ms. Thompson during core instruction and is responsible for providing intensive instructional support for students who have the most difficulty in meeting core standards. Mr. Hart also provides input to the teams for Grades 3–5 regarding supplemental instructional support for students receiving exceptional student education (ESE) services.

In Ms. Thompson's class, the morning begins with a 60-minute block of core mathematics instruction, followed by a 120-minute reading or English language arts block. During an additional 50-minute block after lunch, all students engage in some type of supplemental or intensive instruction or enrichment for reading, mathematics, or both. Math standards in this state are closely aligned with the Common Core State Standards (CCSS).

IDENTIFY AND UNDERSTAND THE MATHEMATICS

For the *Identify and Understand the Mathematics* component, you will read how Ms. Thompson and Mr. Hart prepared themselves to teach the mathematical content. In essence, we pull back the curtain to show you the behind-the-scenes preparation that equips teachers with the mathematical understanding necessary for this process. Ms. Thompson and Mr. Hart go through three stages for this first component. First, they identify the relevant mathematics standards. Second, they look for and learn from an available trajectory that describes how students progress through various stages of learning related to the standards. Third, they consider the role mathematical practices have in the learning process with respect to the identified content.

Math Standard

Ms. Thompson and Mr. Hart's first task is to identify the content that they will be teaching. Based on the curriculum map for fifth graders in their district, they are planning to teach a unit on multiplying multi-digit whole numbers using the standard algorithm. The relevant CCSS standard for fifth grade is as follows (National Governors Association [NGA] Center for Best Practices & Council of Chief State School Officers [CCSSO], 2010).

CCSS.MATH.CONTENT.5.NBT.B.5

Fluently multiply multi-digit whole numbers using the standard algorithm.

In thinking about teaching this standard, Ms. Thompson knows that she and Mr. Hart need to unpack the included content. She also knows they need to consider how this content connects to what students have previously been exposed to in fifth grade as well as in earlier grades in order to ensure students have the prerequisite knowledge to engage successfully with this content. Keeping in mind where their students will be headed in future mathematics lessons, Ms. Thompson and Mr. Hart also look to related standards, both within the grade level and beyond, to help them be purposeful in making decisions about instructional tasks and about how to leverage students' current mathematical ideas.

Ms. Thompson thinks about how these standards connect to other fifth-grade standards, including what students have already been exposed to this year and what they will be expected to learn later in the year, and she shares her thoughts with Mr. Hart. With respect to Number and Operations in Base Ten, their students have worked on the following standard to further their understanding of the place value system (NGA Center for Best Practices & CCSSO, 2010):

CCSS.MATH.CONTENT.5.NBT.A.1

Recognize that in a multi-digit number, a digit in one place represents 10 times as much as it represents in the place to its right and 1/10 of what it represents in the place to its left.

Ms. Thompson notes that the remaining fifth-grade standards for Number and Operations in Base Ten are related to two other mathematical ideas: 1) developing division strategies with whole numbers and 2) developing the four operations (addition, subtraction, multiplication, and division) with decimals to hundredths using place-value ideas, properties of operations, and relationships between the various operations.

Ms. Thompson and Mr. Hart look at what their students were exposed to in fourth grade related to multiplication of multi-digit numbers. In particular, they consider the following fourth-grade CCSS standard (NGA Center for Best Practices & CCSSO, 2010):

CCSS.MATH.CONTENT.4.NBT.B.5

Multiply a whole number of up to four digits by a one-digit whole number, and multiply two two-digit numbers, using strategies based on place value and the properties of operations. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

Ms. Thompson knows the standard algorithm for multi-digit multiplication is one of the most difficult algorithms; students are very error prone when using this algorithm, especially when they have not had ample opportunities to work with multiplication strategies based on place-value concepts and representations such as area models. Both Ms. Thompson and Mr. Hart are aware of the importance of developing students' procedural knowledge from conceptual understanding (NCTM, 2014b). This leads them to recognize that before they work to help students develop the standard multi-digit multiplication algorithm, they will need to revisit this related fourth-grade standard to ensure students have developed a rigorous conceptual understanding of multiplication.

With the insights Ms. Thompson and Mr. Hart have developed by considering not only the target standard but also how it relates to other standards from the previous and current grade levels, they are establishing a better sense about how to build on students' prior knowledge to understand and apply the new fifth-grade multiplication standard. Mr. Hart appreciates the way Ms. Thompson goes deeper in thinking about the content and related learning intentions they have for their students, including connecting the math their students will be learning to that which they have already experienced. This helps Mr. Hart think about what prerequisite content, both at the grade level and below it, he will need to emphasize when he provides more intensive instruction to his students. He knows this content must connect to core math standards.

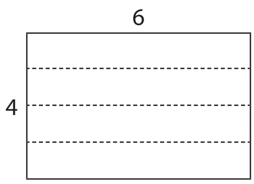
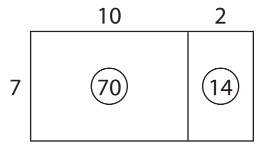
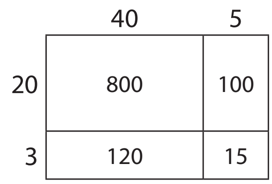
Related Learning Trajectory

Ms. Thompson knows the usefulness of learning trajectories in providing a road map for how student thinking progresses and how to sequence learning experiences to maximize learning of a mathematical concept or skill. She searches online for multiplication trajectories and finds one that describes a progression of students' multiplicative reasoning and strategies. Table E.1 includes the more sophisticated ways of reasoning multiplicatively in this learning trajectory, more likely to be exhibited by older elementary students. (For the full trajectory, see the section on multiplicative reasoning in Chapter 3.) Ms. Thompson is also aware of several learning trajectories she can find online when she is working on other mathematical concepts, such as https://www.turnonccmath.net and http://www.numeracycontinuum.com/continuum-chart.

Table E.1. The upper levels of multiplicative reasoning demonstrated by students

Level 3: Transitional multiplicative strategies (see Figure E.1a)	At this level, students demonstrate an increasingly robust capability of reasoning with multiples as their use of groups becomes more sophisticated. They no longer have to count each group by ones. Strategies such as using area models and open arrays are used to reason through multiplicative situations.
Level 3.3: Repeated abstract composite grouping	Students are aware that a number can be both composite and unitary at the same time, but at this level, students can only think of one of the numbers in a multiplication situation (one of the factors) in this way. For example, with 3×4, students are able to consider the 4 as both a composite unit and unitary at the same time, but they only think of the 3 as unitary—as a way to count the number of fours. They can see the 4 as consisting of 4 single units (unitary) but can also see (or make sense of) the 4 as one "thing" (a composite unit). For 3×4, they would reason $4 + 4 + 4$ (4 three times).
Level 4: Multiplicative strategies (see Figure E.1b)	At this point, students can reason about multiplication and division using the more sophisticated strategies that rely primarily on numerical representations such as partial products, the distributive property, and doubling and halving of quantities.
Level 4.1: Multiplication and division as operations	At this level, students can coordinate two composite units in the context of multiplication or division. For example, with a task such as six groups of four, the student is aware of both 6 and 4 as abstract composite units. The 6 can be used as a count of the groups of four but can also be considered its own composite unit. As a consequence, the commutative property of multiplication makes sense to the student. The student is able to immediately recall and quickly derive many of the basic facts for multiplication and division.

a

Later transitional strategies →		
Area model (less reliant on needing to see every square unit)	Open area model	Open area model
$4 \times 6 = 24$ 	$7 \times 12 = 70 + 14 = 84$ 	$23 \times 45 = 800 +$ $120 + 100 + 15 = 1035$ 

b

Multiplicative strategies →		
Known or derived facts $8 \times 6 = 48$ because $4 \times 6 = 24$ and we need to double that	Commutative property $6 \times 8 = 8 \times 6$	Powers of 10 $3 \times 500 = 3 \times 50 \times 10$ $= 3 \times 5 \times 10 \times 10$ $= 15 \times 100$ $= 1500$
Associative property $(4 \times 6) \times 5$ $= 4 \times (6 \times 5)$ $= 4 \times 30$ $= 120$	Doubling and halving $18 \times 3 = 9 \times 6 = 54$ (9 ↑ half of 18; 6 ↑ 3 doubled)	Distributive property $8 \times 12 = 8(10 + 2)$ $= 8(10) + 8(2)$ $= 80 + 16$ $= 96$
Partial products 23 × 45 15 (5 × 3) 100 (5 × 20) 120 (40 × 3) + 800 (40 × 20) 1035	Standard algorithm 1 ~~1~~ 23 × 45 115 + 920 1035	

Figure E.1. Examples of strategies students use to engage in multiplicative reasoning: later transitional strategies (a) and multiplicative strategies (b).

From studying this trajectory, Ms. Thompson and Mr. Hart come to understand that students who are ready to develop the standard algorithm for multi-digit multiplication should be using the later (more developed) multiplicative strategies seen in Level 4 of the learning trajectory, which are based on the various properties and on strategies such as doubling and halving (see Figure E.1B). They should also already be proficient in using the partial products algorithm. Students who are not there yet may be using the later transitional strategies that rely on an area model, or they may just be developing proficiency with the partial products algorithm. Some may also exhibit even less sophisticated reasoning that relies on skip counting or inefficient additive strategies. (See Chapter 3 for information pertaining to these lower levels of reasoning.) Ms. Thompson and Mr. Hart decide to use this learning trajectory to help them identify the level of sophistication of their students' reasoning. For students whose assessment results indicate less sophisticated reasoning, interventions will need to be used.

Math Practices

Ms. Thompson and Mr. Hart remember that their students not only need to learn the math content within the target standards but also need to learn how to meaningfully engage with this content in different ways through the Common Core Eight Standards for Mathematical Practice (see Chapters 2 and 7). Textbox E.1 shows these practices as a reference.

As Ms. Thompson thinks about each of the math practices, she realizes many could be appropriately utilized in conjunction with the target standard. To make things more manageable, she identifies two she wants to explicitly emphasize within her instruction. (These practices are bold in Textbox E.1.) Ms. Thompson determines that one practice, *Look for and make use of structure* (NGA Center for Best Practices & CCSSO, 2010), fits well because her students will be learning to multiply multi-digit whole numbers by relating their understanding of area models, the distributive property, and place value to the standard algorithm. She chooses a second practice, *Construct viable arguments and critique the reasoning of others* (NGA Center for Best Practices & CCSSO, 2010), because she wants to help her students

Textbox E.1. Common Core Eight Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. **Construct viable arguments and critique the reasoning of others.**
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. **Look for and make use of structure.**
8. Look for and express regularity in repeated reasoning.

From Common Core State Standards Initiative. (2010). *Common Core State Standards for mathematics.* Retrieved from http://www.corestandards .org/assets/CCSSI_Math%20Standards.pdf; reprinted by permission.

become more comfortable and proficient with engaging in discourse, appropriately listening to and critiquing their peers' arguments and reasoning.

Because the standard multiplication algorithm requires a level of precision in order to ensure the proper digits are multiplied together and recorded appropriately, Mr. Hart suggests that they also emphasize with their students the practice of *Attend to precision* (NGA Center for Best Practices & CCSSO, 2010) when they write down the procedure. Mr. Hart knows this will be an important point of emphasis for several students who tend to be more impulsive and struggle with self-regulation strategies related to organization and time management during independent math work.

CONTINUOUSLY ASSESS STUDENTS

For the *Continuously Assess Students* component, Ms. Thompson and Mr. Hart work collaboratively to assess students' specific learning needs related to the identified content standards and mathematical practices. First, Ms. Thompson reviews students' available benchmark assessment data, using them to project which areas related to multiplication of whole numbers might require further formative informal assessment. Then, Ms. Thompson and Mr. Hart create several assessment tasks to get at these areas. Last, they reflect on their students and possible ways to engage them in responding to the assessment tasks so that they can best determine what their students know, don't know, and why.

Review Available Benchmark Assessment Data

Ms. Thompson and Mr. Hart's school district collects grade-level benchmark data three times during the school year. The multiple-choice benchmark assessments relate directly to the state standards and the end-of-year high-stakes test. Each math benchmark assessment is completed online and evaluates where students are in relation to the standards they have been exposed to when the benchmark assessment is administered (early September, early December, and mid-February). Each typically takes approximately 45–60 minutes to complete. School personnel use the first benchmark assessment in early September to make initial decisions about supplemental and intensive tiered instruction.

The school also utilizes a commercial online curriculum-based measurement (CBM) progress monitoring tool that targets particular grade-level math concepts and skills. These measures are administered more often than the benchmark assessments—every 4 weeks to all students. They target a more specific subset of grade-level, CCSS domain–specific math concepts and skills (determined by grade-level teams). Each CBM assessment includes approximately 20–25 multiple-choice items and typically takes students 30 minutes to complete. Students receiving supplemental or intensive math instruction in addition to core instruction are administered shorter, more focused, and more frequent CBM progress monitoring probes as needed during their supplemental and intensive instructional time.

Given that it is late October, Ms. Thompson has data from the beginning year benchmark assessment and two CBM assessments for all students in her class. The school's first benchmark assessment focused primarily on essential fourth-grade concepts and skills that are prerequisites for success in fifth grade. Ms. Thompson

	Most	Some	Few
Appears to know or understand	• *Has a conceptual understanding of multiplication as equal groups* • *Can use repeated addition to represent multiplication* • *Uses known multiplication facts to derive unknown facts* • *Can use an open area model to represent double-digit by double-digit multiplication and decomposes the tens and ones in the model* • *Can explain and appropriately utilize the commutative, associative, and distributive properties* • *Uses partial products (without a visual model) to solve double-digit multiplication problems*	• *Uses the powers of 10 when appropriate to solve double-digit multiplication problems*	• *Flexibly uses strategies such as doubling and halving to solve double-digit multiplication problems* • *Uses partial products (with a visual model) to solve double-digit multiplication problems* • *Uses an area model that shows all the square units to represent double-digit by double-digit multiplication and decomposes the tens and ones in the model*

Figure E.2. Ms. Thompson's summary of what her 22 students know, based on benchmark and curriculum-based measurement data through late October and organized by most students (80% +), some students (10%–15%), and few students (5% or less).

reviews benchmark results related to prerequisites for the target math content. For her students currently receiving supplemental and intensive math support, she also has access to the progress monitoring data generated more frequently.

Based on these data, Ms. Thompson creates a simple chart (see Figure E.2) to help her visualize what her students appear both to know and not know with respect to key prerequisite concepts and skills related to whole number multiplication.

Assessment Tasks

Ms. Thompson and Mr. Hart utilize the information from Ms. Thompson's benchmark and CBM summary table to think about which kinds of formative assessment to create. They aim to get a more in-depth understanding of their students' thinking and level of understanding with respect to prerequisite concepts and skills that are foundational to multi-digit multiplication. Based on the benchmark assessment data, Ms. Thompson and Mr. Hart determine that the formative assessment should focus on three primary areas: flexible use of multiplicative strategies such as doubling, appropriate use of the properties of multiplication (commutative, associative, distributive), and accurate use of the partial products algorithm (numerical use of partial products). They agree it would be most efficient to develop a short assessment probe addressing these concepts and skills, to be taken by all students.

The two teachers want to be sure that there are multiple tasks for each area of focus so that there are enough items to ensure they get an accurate appraisal of what students know and don't know. (See Chapter 5 for more about developing informal formative assessments.) Furthermore, they want to be sure the tasks assess student engagement in their targeted mathematical practices: 1) *Construct viable arguments and critique the reasoning of others,* 2) *Look for and make use of structure,* and 3) *Attend to precision* (NGA Center for Best Practices & CCSSO, 2010). They decide to include some items that contain word problems (contextualized and applied problems) and some items that do not (see Textbox E.2). To this end, they create six noncontextualized tasks: two requiring knowledge of multiplication properties, two asking students to use flexible multiplication strategies, and

Textbox E.2. Formative assessment created by Ms. Thompson and Mr. Hart to appraise what students know and don't know about prerequisite multiplication concepts and skills

1. Use drawings or manipulatives to demonstrate why $3 \times 4 = 4 \times 3$.
2. Darran thinks $5 \times (2 \times 6)$ is not the same as $(5 \times 2) \times 6$. Please explain why you agree or disagree with Darran.
3. If you did not know how to multiply 5×14, which set of facts would help you find the product? Please explain why you selected a particular answer:

   $5 \times 1 + 5 \times 4$

   $5 \times 10 + 5 \times 4$

   $5 \times 1 \times 4$

   $5 \times 10 \times 4$
4. Xavier thinks that the product of 18×5 is the same as the product of 9×10. Do you agree or disagree with him? Explain why you agree or disagree.
5. Xiao and Maria both used partial products to solve 34×8. Look at their solutions. Explain why you think each solution is correct or incorrect:

   Xiao:
   $$\begin{array}{r} 34 \\ \times\ 8 \\ \hline 32 \\ +\ 240 \\ \hline 272 \end{array}$$

   Maria:
   $$\begin{array}{r} 34 \\ \times\ 8 \\ \hline 32 \\ +\ 24 \\ \hline 56 \end{array}$$
6. Use the partial products algorithm to solve 23×16.
7. Jose's father sells hotdogs at soccer games. There are 12 hotdogs in a pack, and Jose's father goes through exactly 15 packs of hot dogs during one game. How many hot dogs did Jose's father sell during that game? Please explain how you solved this problem. (For students who finish early, ask them to solve using a different multiplication strategy.)
8. Niki collects stamps. She wants to buy an album to hold her stamps. One album holds 12 stamps on a page and contains 22 pages. Another album holds 15 stamps on a page and contains 18 pages. Niki wants to buy the album that holds more stamps. Which one should she buy? Please explain the reasoning behind your answer.

two requiring use of the partial products algorithm. They also include two word problems that require students to apply relevant multiplication strategies. The two teachers estimate that it will take most students between 20 and 30 minutes to complete the assessment.

Student Responses

As students finish, Ms. Thompson and Mr. Hart review students' responses to the informal assessment to get a sense of what they know and what they don't know, and a sense of possible error patterns that may indicate students' misconceptions about important underlying math concepts. Figure E.3 shows Ms. Thompson's and Mr. Hart's summary of their students' responses, which they will use to determine their students' specific mathematical learning needs and then to plan and implement responsive instruction.

DETERMINE STUDENTS' MATH-SPECIFIC LEARNING NEEDS

For the *Determine Students' Math-Specific Learning Needs* component, Ms. Thompson and Mr. Hart use student response data to determine their students' math-specific learning needs. This involves three important activities, which all rely on the evaluation of student responses to the assessment tasks. First, based on the responses, Ms. Thompson identifies students' positions on the related learning trajectory (from the *Identify and Understand the Mathematics* component). She then determines students' misconceptions, as well as what students know (knowledge strengths) and don't know (knowledge gaps) with respect to the identified mathematics. Finally, Ms. Thompson and Mr. Hart determine together which mathematical ideas they need to target specifically that will support students to further develop their mathematical knowledge and skills along the identified learning trajectory and toward the identified standard(s).

Identify Each Student's Position on the Identified Learning Trajectory

Based on students' responses, the majority of students appear to be functioning at Level 4 of the learning trajectory, meaning they can demonstrate reasoning about multiplication using the more sophisticated multiplicative reasoning strategies, which rely primarily on numerical representations that incorporate properties, such as the associative and distributive properties. Most do not rely on the area model to use the partial products algorithm.

Some students have demonstrated a limited understanding of key areas, such as the prompted use of multiplicative strategies such as doubling and halving and reliance on open area models to accurately complete the partial products algorithm. Based on this evidence gathered through assessment, Ms. Thompson judges these students to be functioning at Level 3 of the trajectory.

Two students' assessment performance indicates they are likely functioning at a lower level of the learning trajectory, possibly Level 2, because they need to rely on using an area model showing all the square units to make sense of the partial products algorithm. They demonstrated a solid understanding of the associative and commutative properties but not the distributive property, so they could be making the transition from Level 2 to Level 3.

Teacher: *Ms. Thompson* Class/period: *1st period*			
Areas for the assessment	On target (Write names.)	Limited (Write names.)	Insufficient (Write names.)
Flexible use of multiplicative strategies such as doubling (Level 4) *Task 4*	*Most students*	*Tommy S.* *Jerome B.* *Felisha T.* *NOTES: The students are able to use doubling or halving when prompted but at a very slow pace.*	*Steve A.* *Tamika W.* *NOTES: The students use calculation, not doubling or halving, to determine equivalence (e.g., uses 18 × 5 and 9 × 10, sees they are both 90). The students are unable to use the strategy, even when prompted.*
Appropriate use of the properties of multiplication (commutative, associative, distributive) (Level 4) *Tasks 1, 2, and 3*	*Commutative property:* *All students*		
	Associative property: *All students*		
	Distributive property: *Most students*		*Distributive property:* *Steve A.* *Tamika W.* *NOTES: The students use multiplication instead of addition (e.g., 5 × 14 = 5 × 10 × 4).*
Accurate use of the partial products algorithm (numerical use of partial products) (Level 4) *Tasks 5, 6, 7, and 8*	*Most students*	*Tommy S.* *Jerome B.* *Felisha T.* *NOTES: The students are able to recognize when partial products are used correctly (Task 5) but not able to use the strategy accurately without relying on an open area model (with and without contexts) (Tasks 6, 7, and 8).*	*Steve A.* *Tamika W.* *NOTES: The students are unable to accurately identify or use the partial products algorithm, even when using an open area model. When Mr. Hart sketched an open area model for Task 5, these students seemed confused as to what the dimensions of the area model represented. They wanted to put those numbers inside the area model, not along the edges. They may need to rely on an area model that shows every square unit.*

Figure E.3. Summary of students' responses to assessment tasks that provides a sense of what they know, what they don't know, and possible error patterns that may indicate students' misconceptions.

Evaluate Prior Knowledge, Strengths and Gaps, and Misconceptions

Most students indicated a readiness for learning the standard algorithm for multi-digit multiplication because they could reason using the partial products algorithm in a purely numerical representation and explain their reasoning. They could also demonstrate a flexible use of multiplicative strategies, including doubling and halving, as well as the associative, commutative, and distributive properties of multiplication.

Three students could appropriately use the properties of multiplication but still relied on open area models to support their reasoning with the partial products algorithm. They also struggled with using and knowing when to use a doubling and halving strategy. It appears these students rely heavily on decomposing multi-digit numbers based on place value, so they are not looking for alternative number relationships to exploit. Although this strategy will help enhance their number and computation sense, it should not hinder their progress toward becoming fluent with the partial products algorithm and the standard algorithm for multi-digit multiplication.

Two students demonstrated appropriate use of the associative and commutative properties, but they demonstrated issues with the distributive property as well as the partial products algorithm, even when supported with open area models. Based on their confusion about what the numbers represented when Mr. Hart used an open area model to illustrate the partial products, it appears they may have some knowledge gaps in the concept of area. These two students also struggled with knowing when and how to use doubling and halving strategies. For all five of these students, Ms. Thompson and Mr. Hart noted that they want to think about how to address the different knowledge and skill gaps as they plan and implement instruction.

Target Math Ideas for Instruction

For most of the students, Ms. Thompson and Mr. Hart will target the mathematical ideas closely associated with the identified math standard: *Fluently multiply multi-digit whole numbers using the standard algorithm* (NGA Center for Best Practices & CCSSO, 2010). In particular, they want to reinforce the relationship between the partial products algorithm, the area model, and the standard algorithm. Embedded in these three constructs is the important idea of the distributive property. Two students in particular, Steve A. and Tamika W., demonstrated insufficient evidence of appropriate use of the distributive property, so Ms. Thompson and Mr. Hart make note that they need to include this property as a target math idea for these students. Given the significance of this property to multiplication, they decide to also include it as a target idea for whole-class instruction. In addition, Ms. Thompson and Mr. Hart note that they will need to help Steve A. and Tamika W. enhance their understanding of area before they can be expected to work productively on multiplication in general.

DETERMINE STRUGGLING LEARNERS' SPECIFIC LEARNING NEEDS

For the *Determine Struggling Learners' Specific Learning Needs* component, Ms. Thompson and Mr. Hart work to determine the kinds of barriers that are likely affecting their struggling learners and making learning mathematics difficult. They first identify the mathematics performance traits they have observed with their struggling learners, then focus on determining the possible learning characteristics and curriculum factor barriers that may be contributing to these performance traits.

Identify Observed Performance Traits

Using the form illustrated in Figure E.4, Ms. Thompson and Mr. Hart note the performance traits they have observed with most or some students in the first period class or with individual students. They note that most students demonstrate knowledge and skills for some math domains and not others or for certain standards within particular math domains and not others. Therefore, for that performance trait, they check off the box labeled Most. They also note that certain groups of students have consistently demonstrated two other performance traits, "The student is able to compute or engages in problem solving accurately but at a very slow pace" and "The student avoids engaging in certain mathematical tasks." In the Some column, they write these students' names in the boxes next to these two performance traits. Finally, Ms. Thompson and Mr. Hart also note individual students (two or fewer) who demonstrated still other performance traits (e.g., "The student demonstrates faulty mathematical thinking or ineffective strategies

Teacher: *Ms. Thompson and Mr. Hart* Class/period: *1st period*			
Mathematics performance traits	Most (✓)	Some (Write names.)	Individual (Write names.)
The student demonstrates knowledge and skill for some mathematical domains and not others, or for certain standards within a domain and not others.	✓		
The student demonstrates faulty mathematical thinking or ineffective strategies when problem solving.			*Steve A.* *Tamika W.*
The student is able to compute or engages in problem solving accurately but at a very slow pace.		*Tommy S.* *Jerome B.* *Felisha T.*	
The student has difficulty with generalizing knowledge and skills to other mathematical concepts, skills, and contexts.			*Tommy S.* *Jerome B.*
The student demonstrates mathematical abilities at one point in time but then is unable to demonstrate the same abilities later.			*Steve A.* *Jerome B.*
The student avoids engaging in certain mathematical tasks.		*Tommy S.* *Jerome B.* *Steve A.* *Tamika W.*	

Figure E.4. One way to record students' math performance by most, some, and individual.

when problem solving," "The student has difficulty with generalizing knowledge and skills to other mathematical concepts, skills, and contexts," "The student demonstrates mathematical abilities at one point in time but then is unable to demonstrate the same abilities later"). In the Individual column, they write these students' names in the boxes for these learning traits.

In reviewing their notes, the two co-teachers develop a picture of the kinds of math performance traits demonstrated by most of their students, by some students, and by only one or two students. Ms. Thompson and Mr. Hart can also easily see which students are demonstrating more math performance trait difficulties than others. This information now provides them with reference points to begin thinking about what potential learning characteristic and curriculum factor barriers they will want to consider when planning and implementing instruction.

In considering potential learning characteristic barriers, Ms. Thompson and Mr. Hart think about their students as they review their notes on performance traits (see Figure E.4). Four of Ms. Thompson's students have identified disabilities and therefore have individualized education plans (IEPs): Steve A., Tamika W., Tommy S., and Jerome B. In addition, one student identified with ADHD, Felisha T., has a Section 504 accommodation plan. Each of these students is listed in either the Some or Individual column (see Figure E.4) as demonstrating more than one performance trait. Ms. Thompson and Mr. Hart know that, for these five students, they will need to consider all the typical learning characteristics of struggling learners when thinking about how these characteristics might be associated with their performance traits.

Ms. Thompson finds it helpful to have Mr. Hart as a collaborator because he helps her to better understand the disability-related needs of her students with identified disabilities: Steve A., Tamika W., Tommy S., and Jerome B. Tamika W. also has an identified speech-language impairment, related to difficulties articulating particular sounds when she speaks. Tommy S. and Jerome B. also are identified as having ADHD—Tommy S. with the primarily inattentive type (distractibility) and Jerome B. with the combined type (inattention and impulsivity/hyperactivity). Felisha T., who does not have an IEP but has a 504 accommodation plan, is, like Tommy S., identified as having the primarily inattentive type of ADHD. Ms. Thompson and Mr. Hart decide to create a table that shows the students with identified disabilities, their particular disabilities, and specific information about particular cognitive, social-emotional, and behavioral issues—as documented in the students' IEP and cumulative folders, and also based on the connections Mr. Hart made based on his knowledge of disability-related learning needs. Table E.2 shows the teachers' notes.

With this information at hand, Ms. Thompson and Mr. Hart can now start thinking about what learning characteristics could be contributing to their students' math performance traits. They go back to their "most, some, individual" notes (see Figure E.4) and consider each student, the performance trait, and the information gathered for each student with an identified disability (see Table E.2). For example, they note that Steve A. demonstrates three performance traits (in addition to the one most of Ms. Thompson's students demonstrate). For each of Steve A.'s performance traits, the two teachers consider his cognitive, social-emotional, and behavioral needs related to his disability and which learning characteristics are likely contributing to his performance trait difficulties. Table E.3 shows an example of their thinking.

Table E.2. Ms. Thompson's and Mr. Hart's notes about their students with identified disabilities

Student	Identified disabilities	Cognitive, social-emotional, behavioral issues
Steve A.	*Learning disabilities*	*Cognitive—difficulty connecting more than one idea to another, difficulty applying learning strategies, memory retrieval impairments*
Tamika W.	*Learning disabilities* *Speech-language impairment—articulation*	*Cognitive—difficulty connecting more than one idea to another, difficulty applying learning strategies, memory retrieval impairments* *Social-emotional—feels inferior to peers because of her speech articulation difficulties*
Tommy S.	*Learning disabilities* *Attention-deficit/hyperactivity disorder (ADHD)–primarily inattentive type*	*Cognitive—difficulty connecting more than one idea to another, difficulty applying learning strategies, auditory processing difficulties (e.g., slower processing rate with verbal instructions and directions), distractibility*
Jerome B.	*Learning disabilities* *ADHD–combined type (inattention and hyperactivity/impulsivity)*	*Cognitive—difficulty connecting more than one idea to another, difficulty applying learning strategies, memory retrieval and working memory impairments, sometimes responds to verbal directions before he understands what is being asked of him* *Behavioral—needs to move when engaged in seatwork*
Felisha T.	*ADHD–primarily inattentive type*	*Cognitive—distractibility*

Table E.3. Example of Ms. Thompson's thinking about Steve A., his performance traits, and potential learning characteristic barriers

Student	Performance trait	Potential learning characteristic barriers	Potential curriculum factor barriers
Steve A.	*The student demonstrates faulty mathematical thinking or ineffective strategies when problem solving.*	*Metacognitive thinking disabilities* *Knowledge and skill gaps*	
	The student demonstrates mathematical abilities at one point in time but then is unable to demonstrate the same abilities later.	*Memory disabilities—memory retrieval (and working memory?)*	
	The student avoids engaging in certain mathematical tasks.	*Math anxiety* *Learned helplessness*	

Consider Possible Learning Characteristic Barriers

For the first performance trait, Ms. Thompson and Mr. Hart quickly realize that Steve A.'s difficulties in making connections between certain math ideas and applying effective strategies align closely with one of the learning characteristic barriers, metacognitive thinking disabilities. They also suspect that Steve A. has some gaps in his mathematical knowledge base, confirmed by his low scores on math benchmark testing. Ms. Thompson thinks this could also be a contributing factor. With the second performance trait, Mr. Hart focuses on Steve A.'s memory retrieval difficulties, so he suspects that memory disabilities likely contribute to his pattern of being able to demonstrate knowledge of a concept or skill at one point in time but not at another point. Mr. Hart also wonders whether working memory could be a factor in this; during instruction, Steve A. appears to understand parts of the concept being taught but not others. The teachers note this with a question mark. For the third performance trait, both teachers realize that, given the difficulties Steve A. has, he probably shuts down when confronted with mathematics tasks he does not believe he can complete successfully. Steve A. also often raises his hand for help with math, even when he has the knowledge and skill to complete the task, so they think learned helplessness is potentially playing a role as well.

Ms. Thompson and Mr. Hart use the same process to identify the learning characteristics that are most likely contributing to the difficulties of Tamika W., Tommy S., Jerome B., and Felisha T.

Consider Possible Curriculum Factor Barriers

As Ms. Thompson and Mr. Hart continue to think about each of their struggling learners, they consider the five curriculum factors and how any of these might be barriers to their students' math success and might contribute to the math performance traits they demonstrate. Table E.4 shows their thinking for Steve A. in connection to their notes about potential curriculum barriers.

As Ms. Thompson thinks about the math curriculum she uses, she thinks about how certain characteristics thereof might contribute to her students' difficulties. For the first performance trait, Ms. Thompson reviews a few lessons from the teacher's edition of the math textbook. She notices that although each lesson has a segment related to conceptual understanding, little direct connection is made between the concept and the procedures emphasized during the rest of the lesson. In some ways, these two aspects of the lesson—conceptual understanding and reasoning, and procedural understanding and proficiency—seem to be treated as separate sections. It makes sense to her that this might contribute to Steve A.'s tendency to use inefficient strategies when solving problems; it may also explain why he is sometimes off base in connecting what he is doing to why he is doing it. Therefore, Ms. Thompson suspects that the textbook's limited emphasis on integrating conceptual understanding with procedural proficiency is a curriculum factor barrier for Steve A.

In thinking about the second performance trait, Ms. Thompson considers the memory difficulties Steve A. can experience. This makes her wonder whether the curriculum allows Steve A. to fully store what he learns about a new math concept or skill and have enough opportunities to apply or practice it. This in turn makes

Table E.4. Ms. Thompson's and Mr. Hart's thinking about Steve A., his performance traits, potential learning characteristic barriers, and potential curriculum factor barriers

Student	Performance trait	Potential learning characteristic barrier	Potential curriculum factor barrier
Steve A.	*The student demonstrates faulty mathematical thinking or ineffective strategies when problem solving.*	*Metacognitive thinking disabilities* *Knowledge and skill gaps*	*Level of emphasis placed on the integration of conceptual understanding with procedural proficiency*
	The student demonstrates mathematical abilities at one point in time but then is unable to demonstrate the same abilities later.	*Memory disabilities—memory retrieval (and working memory?)*	*Instructional pacing*
	The student avoids engaging in certain mathematical tasks.	*Math anxiety* *Learned helplessness*	*Lack of utilizing effective mathematics practices for struggling learners across instructional tiers in multi-tiered systems of supports (MTSS)*

her consider whether the instructional pacing is appropriate for Steve A. Given his memory disabilities, the pacing might be too rapid for Steve A. to fully learn and become proficient with newly introduced concepts. He may be able to demonstrate what he understands in the moment, but when he is asked to apply it later in the lesson or on another day, he cannot effectively retrieve this learning from memory because he did not have enough opportunities to apply his learning in order to make retrieval automatic. It is also possible that the lesson's instructional pace was faster than Steve A.'s ability to process the information efficiently, affecting his working memory.

As Ms. Thompson thinks more deeply about Steve A., his performance traits, learning characteristic barriers, and potential curriculum factor barriers, she begins to realize there are some disconnects between the instruction emphasized in the math textbook and his learning needs. Ms. Thompson hypothesizes that this could be a reason for Steve A.'s hesitation to engage in certain mathematics activities: He has not adequately learned them. So, Ms. Thompson notes utilization of effective instructional practices for struggling learners as another important potential curriculum factor barrier for Steve A.

PLAN AND IMPLEMENT RESPONSIVE INSTRUCTION

At this point, Ms. Thompson and Mr. Hart have worked fully through the first four components of the Teaching Mathematics Meaningfully Process. They have integrated the two perspectives illustrated in Figure 12.1—that of a mathematics

Excerpted from Teaching Mathematics Meaningfully: Solutions for Reaching Struggling Learners, Second Edition
by David H. Allsopp, Ph.D.,LouAnn H. Lovin, Ph.D., & Sarah van Ingen, Ph.D.

teacher and that of a special education teacher—as they worked through each aspect of the two related components, *Determine Students' Math-Specific Learning Needs* and *Determine Struggling Learners' Specific Learning Needs.*

For the *Plan and Implement Responsive Instruction* component, you will read how Ms. Thompson and Mr. Hart take what they learned so far and use it to plan and implement their instruction in ways that respond to their struggling learners' needs. This final component of the Teaching Mathematics Meaningfully Process includes the following steps:

- Developing a math instructional hypothesis (see Chapter 5) to guide planning
- Planning for and implementing effective instructional practices (see Chapters 7–8)
- Reflecting and revising instruction based on student performance data (see Chapter 5)

As you read about how Ms. Thompson and Mr. Hart engage in this process, you will learn how they identify instructional hypotheses to address the needs of the students whose formative assessment results indicated knowledge and skill gaps. Also, you will learn how the two teachers plan and implement instruction that is organized at three levels based on how MTSS/RTI is implemented in their school:

- Less intensive (whole-class, differentiated core instruction at Tier 1)
- More intensive (small-group, supplementary instruction at Tier 2)
- Even more intensive (intensive instruction at Tier 3 for a few students)

As noted previously, the school has a daily 50-minute period devoted to providing students with additional instructional time in reading and mathematics, used either for more intensive instruction or for extension and enrichment. During this time, general education teachers, such as Ms. Thompson, provide supplementary instruction (Tier 2) for students who need it; special education teachers, such as Mr. Hart, and a math coach provide even more intensive instruction (Tier 3). Teachers either alternate days for reading and mathematics instruction or split the 50-minute period in half to address each content area.

With this information in mind, we focus on how Ms. Thompson and Mr. Hart plan and implement instruction for struggling learners at each level or tier. Please note the icons in the right margin that highlight how these teachers' instruction incorporates certain practices discussed throughout the book: EIAs (Chapter 7), anchors of instruction that can be intensified within MTSS/RTI (Chapter 10), and the MTPs (Chapter 8).

Instructional Hypothesis

Reflecting on students' responses from the formative assessment tasks, Ms. Thompson and Mr. Hart determine that most have the prerequisite knowledge and skills to achieve the overall learning intention, multiplication of multi-
3 digit numbers using the standard algorithm. However, five students struggle with different prerequisites. Although Ms. Thompson and Mr. Hart do not believe an instructional hypothesis is needed to guide instruction for most of their students,

they decide instructional hypotheses would be helpful for these five. For the three students (Tommy S., Jerome B., and Felisha T.) who cannot accurately use the partial products algorithm without relying on an area model, Ms. Thompson and Mr. Hart develop the following instructional hypothesis to support planning and instruction:

Given two multi-digit numbers to multiply:

Students are able to recognize situations that involve multiplication and accurately use the partial products algorithm when using an open area model.

Students are unable to use the partial products algorithm without relying on an area model

. . . *because* they have difficulty keeping track of the numerical partial products without the visual cues provided with the area model.

For the two students (Steve A. and Tamika W.) who are struggling with the distributive property and partial products algorithm, Ms. Thompson and Mr. Hart develop the following instructional hypothesis:

Given two multi-digit numbers to multiply:

Students are able to use the associative and commutative properties for multiplication.

Students are unable to use the distributive property or partial products algorithm even with an area model

. . . *because* they do not understand the relationship between the numbers in the partial product and their distribution within an area model.

Ms. Thompson and Mr. Hart use these two instructional hypotheses to guide their instructional planning and teaching as they differentiate and intensify their instruction across tiers for these five students.

Planning and Implementation

As Ms. Thompson and Mr. Hart start to plan their instruction based on the two instructional hypotheses, they consider how the intensification of instruction for those who need it aligns with how their school employs MTSS/RTI (see the introductory paragraph for more about this component of the Teaching Mathematics Meaningfully Process). Ms. Thompson and Mr. Hart begin to plan a sequence of teaching and learning activities that will be used several days. They keep in mind that, because the students have already had lots of experiences using base-ten materials and area models to think about multiplication and have connected these ideas to partial products, the primary goal is for students to develop the written
6 record for the standard algorithm for multi-digit multiplication. Because most of these students are proficient with the partial products algorithm, Ms. Thompson and Mr. Hart agree that instruction related to the target standard should begin by connecting the two algorithms, partial products and standard, using concrete or representational (i.e., semi-concrete, such as an area model) models first.
8 They decide to focus the first few lessons on using area models in conjunction
1 with the written record. Once students are able to demonstrate and articulate

4 an understanding of what is happening with the models, they will move to the written algorithm without relying on the area model. They decide to start with a word problem to reinforce how multiplying multi-digit numbers applies to the real world.

After working with both the standards and the learning trajectory, Ms. Thompson and Mr. Hart articulate the following overall and long-term learn-
2 ing intentions for their students as follows: students will be able to (1) conceptually understand the mathematical ideas related to multiplying multi-digit numbers
1 using the standard algorithm, (2) multiply multi-digit whole numbers with proficiency, and (3) make sense of and solve word problems that involve multi-digit multiplication of whole numbers.

Differentiated Whole-Class Core Instruction (Less Intensive, Tier 1)

Both Ms. Thompson and Mr. Hart agree that using a systematic instruction
1 approach is important for all their students. In particular, it will allow them to evaluate student learning after each lesson and determine what to emphasize in the next. Based on the assessment data gathered, Ms. Thompson and Mr. Hart decide to use parallel tasks, one with smaller numbers and one with larger num-
1 bers, during the initial whole-class lesson. Parallel teaching, in which two teachers teach two different groups the same content, at the same time, in differentiated ways, is a co-teaching model that supports differentiating instruction within whole-class or large-group contexts (Friend & Cook, 1996). Doing this will lower
7 the teacher–student ratio for those students who are struggling with the mathematics content. For those struggling with the distributive property and the area model, the task with a single-digit number multiplied by a double-digit number will reduce the number of partial products on which they must focus. For students with working memory difficulties, such as Steve A., this strategy will lessen the cognitive load necessary to complete the task—making it more likely that they will be able to process the numbers accurately and complete the necessary cognitive actions. This strategy could also help to alleviate students' potential anxiety about attempting something new. The teachers decide on these word problems as
2 their parallel tasks:

- In the front section of the school auditorium are 6 rows where 45 students can sit in each row. How many students can sit in the front of the auditorium?
- In the front section of the school auditorium are 23 rows where 45 students can sit in each row. How many students can sit in the front of the auditorium?

Students are asked to draw the corresponding rectangular area, mark off the area that aligns with each of the partial products, and then complete a written record of the multiplication, using the partial products on a recording sheet that
3 has base-ten columns identified (see Figure E.5). Ms. Thompson and Mr. Hart provide base-ten grid paper (see Figure E.6) to Steve A. and Tamika W. This paper not
8 only explicitly shows the individual squares (as opposed to an open-area model where these squares are implied), but also organizes the squares into groups of 10 rows and 10 columns. This structure is intended to eliminate the need to count all the individual squares and also supports students' partitioning the numbers being multiplied into their respective place values.

Th	H	T	O
		2	7
	×	4	5
	1	3	5
	1	0	0
	2	8	0
+	8	0	0
1	2	1	5

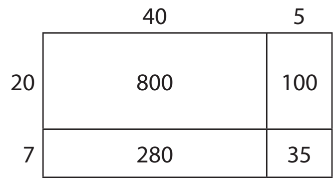


Figure E.5. Place value recording sheet and an open area model showing the partial products for 27 × 45.

At this point in the lesson, it is time to introduce the written record for the standard algorithm. Ms. Thompson and Mr. Hart recognize that based on students' learning needs, some students will need less guidance and others will need more. For those students needing less guidance, their task is to consider the written record they have created using partial products and another one given to them that uses the standard algorithm along with the corresponding area model. Together, they will work to figure out the written record of the standard algorithm (i.e., what the numbers mean and where they came from, and why numbers are recording in particular places). Ms. Thompson and Mr. Hart provide these students with a sequence of written records, as seen in Figure E.7, so the students can see how and when numbers are introduced into the written record.

Ms. Thompson and Mr. Hart know they need to offer more support to some students by starting with a simpler problem and being more explicit. For these students, they use the first word problem that results in 6 × 45. After students have completed the partial products algorithm and corresponding area model for the computation (see Figure E.8), Mr. Hart works with the small group, asking a series of focused questions to help them relate the two written records, such as the following:

Where is the 30 in the partial products and in the area model? What numbers were used to get 30? Where and how is the 30 recorded in the standard algorithm? Why is the 3 recorded above the 40 in the 45?

He knows he needs to use strategies such as visual cuing to help students make these connections (see Figure E.9). Mr. Hart highlights the three tens in the partial products algorithm and uses an arrow to show the connection to the regrouped three tens in the standard algorithm. The use of visuals to help students make connections between mathematical ideas is an effective instructional practice for struggling learners because it helps students focus on important features of mathematical ideas and tasks despite attention difficulties or cognitive processing impairments. Mr. Hart works with this small group of students on additional multiplication of single-digit numbers by double-digit numbers, eventually leading to multiplication problems involving two double-digit numbers.

Gradually scaffolding content in order to help students succeed, with less difficult content expectations initially and then with more difficult expectations later on, supports struggling learners to be willing to take risks. This reduces the likelihood

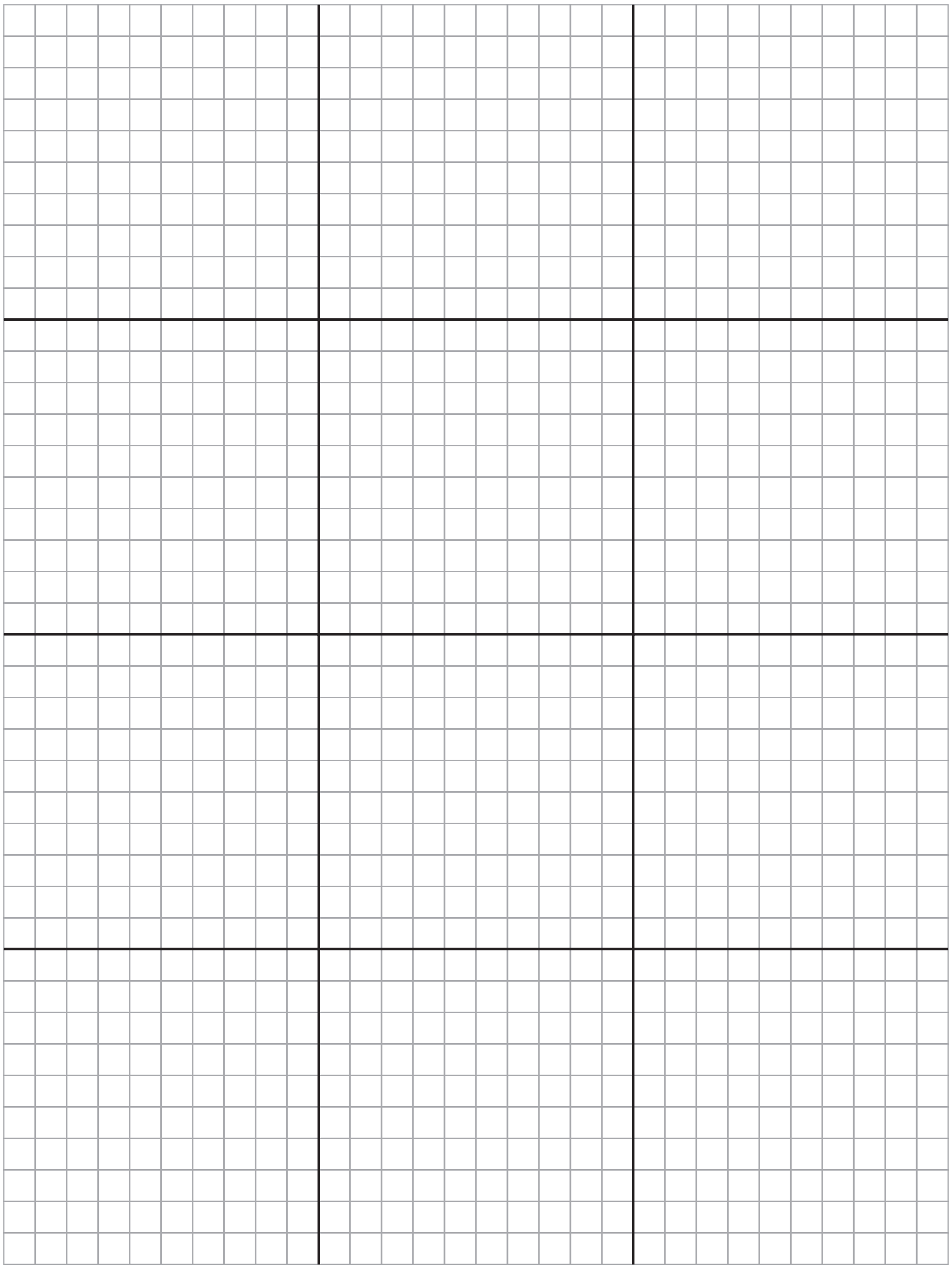


Figure E.6. Base-ten grid paper. (Various open source sheets can be found via an Internet browser search.)

that they will engage in learned helplessness. This practice also supports the needs of students who process information more slowly so they can become proficient with less demanding mathematics tasks before moving to more demanding tasks.

Ms. Thompson and Mr. Hart are also sensitive to the idea that with both the partial products and the standard algorithm, it is imperative that students still

Step 1

Th	H	T	O
		3	
		2	7
	×	4	5
	1	3	5

Step 2

Th	H	T	O
		2 3	
		2	7
	×	4	5
	1	3	5
1	0	8	0

Step 3

Th	H	T	O
		2 3	
		2	7
	×	4	5
	[1]1	3	5
+ 1	0	8	0
1	2	1	5

Figure E.7. Place value recording sheet showing the steps to the standard algorithm for 23 × 45.

5 be precise in the language used. For example, when students who are finding the product of 23 × 45 multiply the 2 and 5, they should say "20 times 5" so that it is more apparent to the student why he or she writes the 1 in the hundreds place.

Supplemental Small-Group Instruction (More Intensive, Tier 2)

The three students who currently receive supplemental mathematics instruction—Tommy S., Jerome B, and Felisha T.—demonstrated they were ready to move toward understanding and becoming proficient with the standard algorithm for multiplication. For this reason, Ms. Thompson and Mr. Hart agree that, although they will likely need some additional instruction on concepts and skills covered during whole-class core instruction, they mostly will benefit from additional response and practice opportunities in order to become proficient with these concepts and skills. Ms. Thompson decides to begin each supplemental instruction session at a small-group table by engaging students in a pre-instructional "check" activity, in which she presents several prompts related to these core concepts and skills and asks
2 students to quickly respond on individual dry-erase boards. As students respond, Ms. Thompson notes any error patterns or apparent misconceptions and writes them in a small journal she uses to informally track students' progress during supplemental instruction. (She finds using a supplemental instruction journal this way helps her to efficiently plan from day to day and pinpoint where to target instruction.) Based on her observations during this pre-instructional check, Ms. Thompson then determines which content or mathematical practices her students need more support in understanding. She communicates to students her learning intentions for the session and how these relate to what she observed during the pre-instructional
2 check.

Th	H	T	O
		4	5
	×		6
		3	0
+	2	4	0
	2	7	0

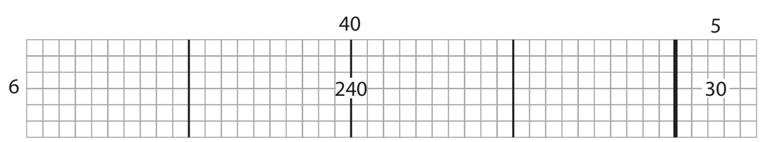


Figure E.8. Place value recording sheet and an area model with the square units showing the partial products for 6 × 45.

Partial Products				Standard		
H	T	O		H	T	O
	4	5			³4	5
×		6		×		6
	(3)	0		2	7	0
+2	4	0				
2	7	0				

Figure E.9. Place value recording sheet comparing the algorithms for partial products and the standard algorithm for 6 × 45.

Knowing her students and considering the nature of the mathematics content standard, Ms. Thompson decides to emphasize the three additional instructional intensification anchors, *Explicitness and Teacher Direction*, *Teach Math Metacognition*, and *Opportunities to Respond.* She believes that her students will benefit from moderate levels of intensification for the first two of these anchors and a high level of intensification for *Opportunities to Respond.* (This is an example of differentiating intensity among the seven instructional anchors; see Chapter 10 for discussion about differentiating the intensity levels of these within MTSS/RTI.) She decides this because she knows that to build appropriate levels of proficiency, her students will likely need some modeling or reteaching and greater opportunities to respond and practice. She understands that Tommy S., Jerome B., and Felisha T. need more instructional time devoted to responding and practice than possible during whole-class core instruction.

For times when her students need additional modeling or reteaching of concepts or skills to build initial and advanced understanding, Ms. Thompson utilizes EIA #8: *Utilize visuals.* Next to her small-group table, she has a small storage box that contains manipulatives, various graphic organizer ideas, individual dry-erase boards and markers, folder instructional board games, and examples of different ways to draw solutions to different types of equations. Depending on which concept, skill, or practice students need more support in understanding, she uses these materials to help students visualize its important features, repeating the visual cuing practices she and Mr. Hart used during whole-group instruction.

When students need support in recalling steps for completing the standard multiplication algorithm or solving multiplication word problems, Ms. Thompson teaches the use of explicit learning strategies (see Chapter 8 for examples) that support students' memory recall and help them build metacognitive awareness. For example, she noted that when students were not provided the place value recording sheet (see Figures E.5–E.9), some forgot or did not connect the place value of digits they regrouped using the standard algorithm. So, she taught these students the FIND strategy, which helps them identify and remember the place value of digits in multi-digit numbers (see Figure E.10). The FIND strategy helps students independently recreate the place value template that was used in differentiated whole-class core instruction. This in turn allows them to respond independently while reinforcing thinking about the place value of digits when using the standard algorithm. Over time, Ms. Thompson will fade students' use of the FIND strategy as appropriate.

Find the columns (space between digits)

Insert the *t*s.

Name the place values.

Determine the value of each digit.

Th	H	T	O
		1	
		4	2
	×	2	9
	1		
	3	7	8
	8	4	0
1	2	1	8

Students use the FIND Strategy to monitor the value of digits as they complete the standard multiplication algorithm.

Figure E.10. Example of the FIND Strategy (Mercer & Mercer, 2005) being utilized to monitor the place value of digits when completing the standard multiplication algorithm.

To help her students build their levels of proficiency, Ms. Thompson incorporates multiple response opportunities during both instruction and practice. For
6 example, she utilizes the individual white boards to ensure all students in the group respond to her questions and prompts. She incorporates the use of instruc-
5 tional board games (see Chapter 7) for practice.

Individualized Instruction (Even More Intensive, Tier 3)

Two students, Steve A. and Tamika W., exhibited an underdeveloped understanding of the concept of area and its relationship to multiplication. Both have been identified as students who need more intensive math instruction in addition to core instruction. Although Ms. Thompson and Mr. Hart have attempted to differentiate their practice within whole-class planning and instruction to better support struggling learners' needs, they know Steve A. and Tamika W. need even more intensive support compared to the students receiving Tier 1 instruction or supplemental Tier 2 support.

Based on the instructional hypothesis the two teachers developed from their informal assessment, Mr. Hart plans to organize his instruction according to three goals, which he will apply in each daily 50-minute intensive session with Steve A.
1 2 and Tamika W.

First, he knows they have gaps in their knowledge base related to the standard targeted in whole-class core instruction. They will need explicit systematic instruction in related foundational concepts and skills: the area model, the distributive property, and the partial products algorithm. Second, he knows his students will need multiple response opportunities and practice with these concepts and skills to build their proficiency and fluency. Third, Mr. Hart knows that Steve A. and Tamika W. will need pre-instructional support in order to benefit from whole-class core instruction.

Therefore, Mr. Hart organizes each intensive instructional session according to these three areas of focus as follows:

1. During the first 20–25 minutes, he focuses on the foundational ideas: area, distributive property, and the partial products algorithm. He decides to begin with the foundational concept of area and how it relates to making sense of multiplication, ideas with which these students are still struggling.

2. During the next 15–20 minutes, he engages his students in practice opportunities related to one of these foundational concepts.
3. During the last 10 minutes or so, he preteaches content he and Ms. Thompson will cover during the next whole-class core instruction class period or periods. He does this so Steve A. and Tamika W. have a preview of what they will be learning the next day and how it relates to what they are learning during Tier 3 instruction, and they can begin learning the content. Mr. Hart believes doing this will better prepare Steve A. and Tamika W. for the next day's core instruction so that they can engage in the whole-class lesson more successfully.

Area Model Instruction Mr. Hart decides to emphasize the same three instructional intensification anchors used by Ms. Thompson with her supplemental Tier 2 group: *Explicitness and Teacher Direction, Teach Math Metacognition*, and *Opportunities to Respond*. However, in contrast to Ms. Thompson, Mr. Hart greatly intensifies the anchors *Explicitness and Teacher Direction* and *Teach Math Metacognition* in addition to *Opportunities to Respond* (another example of differentiating intensity among the anchors). He begins with single-digit multiplication scenarios, such as the following, and has the students create corresponding rectangular areas on 1 grid paper to represent the scenarios.

Ms. Thompson wants to buy a rug for the classroom that is 5 feet by 6 feet. How much floor space (area) will the rug cover?

Mr. Hart begins with scenarios that are easy to model concretely in the classroom. For this scenario, students can utilize a tape measure, the classroom floor 8 3 tiles that each measure 1 square foot, and 1 foot by 1 foot paper squares to represent, think about, and solve area problems. Mr. Hart first models this using think-10 4 alouds that show his thinking about how much floor space the rug in the scenario will cover. Then, he invites students to do the same and to justify why the area model they created is appropriate and how it accurately represents how much space the rug will cover.

Next, he poses different area scenarios using feet and inches and challenges Steve A. and Tamika W. to determine the different areas using the tape measure and square paper "tiles." He also asks them to mark the individual feet or inches on the paper squares and to justify their response. After each response, the students then represent the same area using their grid paper, labeling the 6 feet or inches and identifying the total area represented on the grid paper inside. Mr. Hart checks both students' responses on the grid paper and provides spe-5 8 cific positive reinforcement and corrective feedback. He asks each student to identify the relationships between the area on the grid and the scenario. Once they are able to identify these relationships, he asks them to write the corresponding multiplication equation.

As the students demonstrate proficiency with scenarios that involve continuous quantities, Mr. Hart begins to use scenarios that involve discrete quantities (i.e., ones students count). This is done to help his students generalize multiplication to discrete quantities using an array structure. For example:

In a classroom, there are 4 rows of 5 desks. How many desks are in the classroom?

Again, Mr. Hart models an example first, using think-alouds, with index cards representing the desks. As he did with continuous quantities, he begins with discrete-quantity scenarios that are easy to model concretely using an array. For example, Steve A. and Tamika W. can use index cards to create the 4 × 5 array. Mr. Hart first models this using think-alouds about how to align the index cards in rows and columns.

To help Steve A. and Tamika W. track the number of each, he marks cards with red and blue highlighters to identify the four rows and five columns. Then, he asks them to do the same and to justify why their array model is appropriate and how it accurately represents the total number of desks in the scenario. Next, he poses different combinations of rows and columns, challenges Steve A. and Tamika W. to work in the same way to create the appropriate array for each, and prompts them to justify their response. After each response, the students represent the same area, using their grid paper in much the same way they did earlier when working with continuous quantities, including writing the multiplication equation underneath.

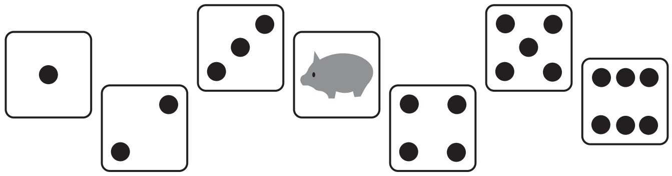
Mr. Hart checks both students' responses on the grid paper and provides specific positive reinforcement and corrective feedback. He asks each student to identify the relationships between the array model with the index cards, the area on the grid, and the equation (e.g., the 4 in the equation is represented by the four rows, the 5 in the equation represents the five columns, the 20 in the equation represents the total number of squares in the area). He does this to make sure that Steve A. and Tamika W. are making the connection between area and multiplication.

Next, Mr. Hart moves to problems that involve the multiplication of a single-digit number by a double-digit number. He addresses these students' misunderstandings about the distributive property by helping them partition into tens and ones the part of the area model corresponding to the double-digit number. He follows a process similar to that used for problems involving single-digit numbers only.

As Mr. Hart works with Steve A. and Tamika W. over time, he takes opportunities to explicitly connect what they are doing to the core content already covered during whole-class instruction because he wants to continuously help his students make these connections. For example, he shows a problem from a prior whole-class session as an example (see Figure E.8) and demonstrates how the single-digit by single-digit area models they have been working with (e.g., 4 × 5) relate to the partial products area model (e.g., 6 × 45) by pointing out how both have rows and columns. In other words, he relates 4 × 5 (four rows of five) to 6 × 45 (six rows of 45).

Practice Because he has only two students and he wants to keep them motivated as they practice, Mr. Hart decides to utilize both instructional games and self-correcting materials (see Chapter 7). For example, he thinks the instructional game Pig would be good for practicing single-digit multiplication of whole numbers (see Figure E.11).

For the Pig game, Mr. Hart decides to have Steve A. and Tamika W. roll two dice to generate two single-digit numbers, represent the area of the resulting rectangle using grid paper, label its rows and columns, identify the area (and product) by writing the number inside the rectangle, and write the corresponding multiplication equation underneath. Mr. Hart monitors and provides feedback and coaching as needed. After the session, Mr. Hart reviews Steve A.'s and Tamika W.'s responses to evaluate their progress and inform planning for the next session.



Purpose: Provides students with computation practice

Materials: Grid paper and pencil; two dice, with a pig sticker on one face of one of the die. (If you do not have a pig sticker, purchase a pink dot sticker from an office supply store and draw a pig face on it.)

Directions: Students work in pairs and take turns. On each turn, a student rolls two dice at the same time. He or she uses the numbers rolled as dimensions of a rectangle. The student draws the rectangle on his or her grid paper. The student multiplies the two numbers and writes the product inside the area of the rectangle. The student keeps a running total of the areas he or she finds. When the pig is rolled, the student has to deduct 20 from his or her total area. Play continues until one student reaches a designated total area (e.g., 100).

Extension 1: Each student rolls three dice at a time to create a one-digit and a two-digit number. Use different colored dice to designate the one-digit number versus the two-digit number. For example, use one white die to generate the one-digit number and use two green dice for the two-digit number. Let the student determine how to use the digits from the dice to create a two-digit number (e.g., 32 or 23). The choice made will provide some insight into the student's number sense and strategy awareness.

Extension 2: Each student rolls four dice at a time to create 2 two-digit numbers. Use two white dice for one two-digit number and two green dice for the other two-digit number. Again, let the student determine how to use the digits from the dice to create a two-digit number.

Figure E.11. Pig Math instructional game. (*Source:* Mercer & Mercer, 2005.)

Pre-instruction for the Next Whole-Class Instruction Session During this portion of more intensive instructional time, Mr. Hart emphasizes the instructional intensification anchor *Explicitness and Teacher Direction*, using an explicit instruction process, LIPP, for connecting what students will learn:

- **L**inking the whole-class learning intention to what Steve A. and Tamika W. are learning about an area model in their even more intensive (Tier 3) session
- **I**dentifying the learning intention they will focus on during subsequent whole-class instruction
- **P**roviding a rationale for why the upcoming learning intention is important and relevant to their lives
- **P**reviewing one or more foundational ideas related to it

Mr. Hart likes the mnemonic LIPP because it helps him remember the four areas to focus on for each pre-instruction session.

Reflect/Revise

Ms. Thompson and Mr. Hart meet regularly to both reflect on and revise their instruction. They do this by evaluating student performance data and through their ongoing observations. This includes students' levels of engagement during instruction; at-the-moment diagnostic interviews and error pattern analyses the teachers complete with students based on their responses during instruction; and other informal formative assessments, such as weekly class mini-quizzes and school-wide continuous progress monitoring data from benchmark and CBM assessments as appropriate. (This is all part of the *Continually Assess Students* component of the Teaching Mathematics Meaningfully Process.)

Ms. Thompson and Mr. Hart concentrate on three questions as they reflect on their instruction across tiers: What is working? What is not working and why? How can we improve what we are doing? Because they continually keep these three questions in mind as they teach, they find that they do not need lots of time when they meet to reflect and revise; reflection becomes a habit of mind, so they already have concrete ideas before they meet. Ms. Thompson and Mr. Hart agree it is wonderful to be able to collaborate for several reasons: They share common perspectives, but each teacher also has another perspective to pull from. Because neither can be present for instruction across all three instructional tiers, collectively they feel that collaboration helps them have a much better handle on all students' performance and related mathematics learning needs, particularly those who are struggling with mathematics. They also find that they are much better prepared when their grade-level team meets to review student performance data to make instructional decisions related to MTSS/RTI at the grade level and for individual students.

χ TAKE ACTION

We designed this case study to illustrate how teachers can integrate the components of the Teaching Mathematics Meaningfully Process. Our illustration is simply one way the process can be carried out; the potential variations and adaptations are limitless and depend on the needs and characteristics of specific students and teachers. Throughout this book, we challenged you to put each component of the decision-making process into action. Our final challenge for you is to integrate the components by putting the entire process into action in your own classroom.

Because of the complexity of this task, you may find it useful to work with a partner and talk through, or write down, how you envision each step playing out in your classroom. Then, go for it! It may feel time consuming and labor intensive, but we encourage you to see the entire process through from beginning to end. Pay close attention to how you and your students respond. Were the results useful? What were the effects on student learning and engagement? Which parts of the process worked well and which need to be tweaked? If you were fortunate enough to have another teacher collaborate with you throughout the process, what were your interactions like? How could they be strengthened in the future? Were there any points of tension or misunderstandings that could be addressed?

Finally, we want to conclude this book by honoring you for the time you have taken to improve your mathematics instruction. Struggling students and students with special education needs historically have not received equitable mathematics instruction. The efforts you have made to read and apply the research-supported strategies in this book are clear indicators of your commitment to meeting the mathematics learning needs of all students. Thank you for this commitment. The outcome—improvements in mathematics teaching and learning for struggling students—is surely worth the effort!

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